ON THE ENUMERATION OF CROSS-SECTIONS AND UNSTABLE VECTOR BUNDLES

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Abstract and introduction

This thesis is an attempt to generalise the methods of two papers [9], [11] by James and Thomas on enumeration problems.

The first five sections consider the problem of enumerating the homotopy classes of cross-sections of a fibration $F \subset X \xrightarrow{p} B$. In §1, we define certain groups $\pi_*^\sigma(p)$ by stabilizing (in a suitable sense) the homotopy groups of the space of cross-sections of p. We obtain generalised 'difference classes' and 'obstruction classes' as elements of $\pi_0^\sigma(p)$ and $\pi_{-1}^\sigma(p)$ respectively, and hence relate these to the enumeration problem.

§2 develops the theory of modified Postnikov towers (MPT's) for fibrations in which the fundamental group of the base acts non-trivially. For this we need techniques for handling k-invariants with local coefficients, but the theory in other respects parallels that of Mahowald [13] and Thomas [23]. An existence theorem is proved in 2.10. §3 introduces the notion of 'stable modified Postnikov tower', which is a crude device to facilitate the setting up of the spectral sequence in the following section. A stable MPT for a fibration kills the stable homotopy groups of the fibre, just as an MPT kills the unstable ones. The existence of stable towers follows from the possibility of 'de-looping' MPT's in the

stable range (3.3). In §4 we set up a spectral sequence (4.19) which, in the simplest case, has the form $E_1^{s,t} \sim H^t(B;\pi_s^\Sigma F) \Rightarrow \pi_*^\sigma(p)$ where $\pi_s^\Sigma F$ is the s'th stable homotopy group of F. This is our main tool for calculating $\pi_*^\sigma(p)$, and for enumerating cross-sections. An essentially similar approach to the enumeration problem has been used (independently) by J.F. McClendon [12]. Our spectral sequence appears to be a formalisation of the last section of [12].

In §5 we compute MPT's for the fibrations $BO_n \to BO$. These extend the calculations of Mahowald [13] for $BSO_n \to BSO$ to the non-orientable case and to the next two homotopy groups of the fibre. Instead of giving the defining relations of the k-invariants, we display the corresponding differentials in our spectral sequence: these contain equivalent information. We apply the results of the computations to determine the number of regular homotopy classes of immersions of real projective n-space P^n in Euclidean space in dimensions near the stable range.

§6 considers the problem of determining the number of real k-plane bundles over a complex X which are stably equivalent to a given bundle. The methods here are developments of those of James and Thomas [9]. We generalise one of the theorems of [9] to non-orientable bundles, and enumerate (n-1)- and

$(n-2)$- dimensional normal bundles to P^n in certain cases.

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§1. The σ-stable homotopy groups of a fibration.

1.0 We use the term fibration to mean a surjective map which has the homotopy lifting property for all spaces. Throughout this section we consider fibrations with a fixed CW-complex B as base. If $p\colon E \to B$ is a fibration, we denote by sec p the space of sections $\{f\colon B \to E \mid pf = 1_B\}$ with the compact-open topology.

1.1 Lemma. Let $(p,p')\colon(E,E') \to (B,B)$ be a pair of fibrations with fibre-pair (F,F'). Suppose the pair (F,F') is k-connected. Then the pair (sec p, sec p') is $(k - \dim B)$-connected.

Proof. Take any map $f\colon(D^r, S^{r-1}) \to (\sec p, \sec p')$ with $r \leq k - \dim B$. Then f is adjoint to a section $\hat{f}$ of the fibration $p \times 1 : E \times D^r \to B \times D^r$ such that $\hat{f}|B \times S^{r-1} \subset E' \times S^{r-1}$. Since the dimension of the base of the fibration-pair $(p \times 1, p' \times 1)\colon(E \times D^r, E' \times D^r) \to B \times D^r$ is not greater than the connectivity of the fibre-pair (F,F'), the section $\hat{f}$ is homotopic rel $B \times S^{r-1}$ to a section of $p' \times 1$. Taking the adjoint, we obtain a homotopy rel S^{r-1} from f to a map $D^r \to \sec p'$. Hence $\pi_r(\sec p, \sec p') = 0$.

1.2 Let $\mathcal{F}_B$ be the category whose objects are fibrations with base B and whose morphisms are fibre-preserving maps. We introduce two functors $\sigma, \lambda\sigma : \mathcal{F}_B \to \mathcal{F}_B$. Suppose $p: E \to B$ is a fibration with fibre F.

<u>The fibration σp.</u> The total space $E(\sigma p)$ is obtained from $E \times I$ by identifying the points (e, t) and (e', t') if $pe = pe'$ and $t = t' \in \partial I$. We give the resulting set the coarsest topology consistent with the continuity of the maps onto I, E given by $(e,t) \to t$ (all (e,t)) and $(e,t) \to e$ $(t \notin \partial I)$, and of $\sigma p : E(\sigma p) \to B$ by $(e,t) \to pe$. Thus σp is homeomorphic to the Whitney join of p with a trivial S^0-bundle.

The fibre ΣF of σp is a retopologised version of the unreduced suspension SF. However, $\Sigma F \simeq SF$ because the map $\phi : \Sigma F \to SF$ given by

$$\phi(e,t) = \begin{cases} (e,0) & t \leq 1/4 \\ (e, 2t - \frac{1}{2}) & 1/4 \leq t \leq 3/4 \\ (e,1) & t \geq 3/4 \end{cases}$$

is a homotopy inverse for $1 : SF \to \Sigma F$.

σp has two sections $s_0, s_1 : B \to E(\sigma p)$ defined by $s_i(b) = (e,i)$ where $e \in p^{-1}b$. We call s_0 the <u>canonical section</u> and use s_0 as base-point in sec σp.

<u>The fibration $\lambda\sigma p$</u>. The total space $E(\lambda\sigma p)$ consists of the set of all paths $\omega : I \to E(\sigma p)$ which satisfy

(i) $\omega(I) \subset$ some fibre of σp

and (ii) $\omega(0) \in s_0(B)$, $\omega(1) \in s_1(B)$.

We give $E(\lambda\sigma p)$ the compact-open topology and define $\lambda\sigma p : E(\lambda\sigma p) \to B$ by $\omega \to p\omega(I)$. Thus the fibre of $\lambda\sigma p$ is the space $\Lambda\Sigma F$ of paths in ΣF from one suspension point to the other.

1.3 We construct a <u>canonical embedding</u> $\psi : E \subset E(\lambda\sigma p)$ by taking the point e to the path $t \to (e,t)$. This gives a natural transformation $1 \to \lambda\sigma$, and an embedding $\sec p \subset \sec \lambda\sigma p$.

1.4 <u>Proposition</u>. The pair $(\sec \lambda\sigma p, \sec p)$ is $(2 \operatorname{conn} F - \dim B + 1)$-connected.

Proof. Consider the fibration-pair $(\lambda\sigma p, p)$. Since $\Sigma F \simeq SF$, Freudenthal's suspension theorem shows that the fibre-pair $(\Lambda\Sigma F, F)$ is $(2 \operatorname{conn} F + 1)$-connected. The result now follows from 1.1.

1.5 <u>Proposition</u>. Suppose p has a section $s : B \to E$. Then $s \in \sec p \subset \sec \lambda\sigma p$ induces isomorphisms

$s_{\#} : \pi_r(\sec \lambda\sigma p; s) \to \pi_{r+1}(\sec \sigma p; s_0)$ for all $r \geq 0$.

Proof. We use s to construct a based homotopy equivalence between sec $\lambda\sigma p$ and the loop space $\Omega(\sec \sigma p;\ s_o)$.

sec $\lambda\sigma p$ is homeomorphic to the path-space $P = (\sec \sigma p, s_o, s_1)^{(I,0,1)}$ under the adjointness relation $[E(\sigma p)^I]^B \approx [E(\sigma p)^B]^I$. Construct a path μ in sec σp by $\mu(t)(b) = (s(b), 1-t)$ for $t \in I$, $b \in B$. Then $\mu(0) = s_1$, $\mu(1) = s_o$. Define a map $P \to \Omega\ (\sec \sigma p;\ s_o)$ by $\omega \to \omega * \mu$, where $*$ denotes track addition. This is a homotopy equivalence, but the composite

$$\sec \lambda\sigma p \approx P \to \Omega(\sec \sigma p;\ s_o)$$

carries the base-point s to the loop $\mu^{-1} * \mu$. Using the obvious contraction of the loop $\mu^{-1} * \mu$, modify the homotopy equivalence to a based one. The result follows.

1.6 Whether p admits sections or not, we can apply the last proposition to the fibrations $\sigma^n p$ for $n > 0$, taking s to be the canonical section s_o. Combining the resulting isomorphism $s_{o\#}$ with the homomorphism induced by the embedding $E(\sigma^n p) \subset E(\lambda\sigma\sigma^n p)$, we obtain a suspension homomorphism σ_* as the composition

$$\pi_r(\sec \sigma^n p;\ s_o) \to \pi_r(\sec \lambda\sigma^{n+1} p;\ s_o) \approx \pi_{r+1}(\sec \sigma^{n+1} p;\ s_o).$$

Since the fibre $\Sigma^n F$ of $\sigma^n p$ has connectivity at least conn $F + n$, 1.4 shows that σ_* is an isomorphism for $r \leq 2$ conn $F + 2n - \dim B$, and an epimorphism for

$r = 2 \operatorname{conn} F + 2n + 1 - \dim B.$

1.7 Definition. Let i be an integer. The i'th σ-stable homotopy group $\pi_i^\sigma(p)$ of the fibration p is the direct limit of the system

$$\cdots \xrightarrow{\sigma_*} \pi_{n+i}(\sec \sigma^n p;\, s_o) \xrightarrow{\sigma_*} \pi_{n+i+1}(\sec \sigma^{n+1} p;\, s_o) \xrightarrow{\sigma_*} \cdots$$

If dim B is finite this sequence consists of isomorphisms from some point onward: indeed $\pi_{n+i}(\sec \sigma^n p;\, s_o) \approx \pi_i^\sigma(p)$ if $n \geqslant i + \dim B - 2 \operatorname{conn} F$.

The following result is the reason for our interest in these groups.

1.8 Theorem. Let $p : E \to B$ be a fibration with fibre F. Suppose that p admits sections and that B is a complex of dimension $\leqslant 2 \operatorname{conn} F$. Then there is a free, transitive action of the abelian group $\pi_o^\sigma(p)$ on the set of homotopy classes of sections of p.

Proof. Since $\dim B \leqslant 2 \operatorname{conn} F + 1$, $\pi_1(\sec \sigma p;\, s_o) \sim \pi_o^\sigma(p)$. The set of homotopy classes of sections of p is $\pi_o(\sec p)$, which maps isomorphically onto $\pi_o(\sec \lambda\sigma p)$ by 1.4 because $\dim B \leqslant 2 \operatorname{conn} F$. Hence it suffices to produce a free transitive action of $\pi_1(\sec \sigma p;\, s_o)$ on $\pi_o(\sec \lambda\sigma p)$. But $\pi_o(\sec \lambda\sigma p)$ is the set of homotopy classes of paths in sec σp which begin at s_o and end at s_1; and π_1 acts freely and transitively on this set by track addition.

1.9 Corollary. If $\dim B \leq 2 \operatorname{conn} F$, the number of homotopy classes of sections of p is either zero or equals the order of $\pi_0^\sigma(p)$.

1.10 If $\dim B \leq 2 \operatorname{conn} F$, we define the difference element $\mathcal{D}(f,g)$ of any two sections f, g of p: $\mathcal{D}(f,g)$ is the unique element of $\pi_0^\sigma(p)$ such that $\mathcal{D}(f,g)\cdot\{g\} = \{f\}$, where the dot denotes the action of 1.8. We deduce from 1.8 : (i) $\mathcal{D}(f,g) + \mathcal{D}(g,h) = \mathcal{D}(f,h)$

(ii) for fixed f and variable g, $\mathcal{D}(f,g)$ takes all values in $\pi_0^\sigma(p)$

(iii) $\mathcal{D}(f,g) = 0$ iff $f \simeq g$.

If $\dim B \leq 2 \operatorname{conn} F + 1$, we define the obstruction element $\mathcal{O}(p) \in \pi_{-1}^\sigma(p)$ as follows. σp satisfies the hypotheses of 1.8, and we set $\mathcal{O}(p) = \mathcal{D}(s_0,s_1) \in \pi_0^\sigma(\sigma p) = \pi_{-1}^\sigma(p)$.

1.11 Theorem. Assume $\dim B \leq 2 \operatorname{conn} F + 1$. Then p has a section iff $\mathcal{O}(p) = 0$.

Proof. If $\mathcal{D}(s_0,s_1) = 0$ then s_0, $s_1 \in \sec p$ are homotopic. Any path joining them gives a section of $\lambda\sigma p$. Hence $\sec \lambda\sigma p$ is non-empty. By 1.4, sec p is also non-empty. The converse is trivial.

This makes it clear that the group $\pi_{-1}^\sigma(p)$ is non-zero in general. Our next task is to set up a spectral sequence for computing the groups $\pi_*^\sigma(p)$.

§2. Modified Postnikov towers with local coefficients

2.0 In this section we shall want to construct maps by methods of obstruction theory. We therefore restrict our attention entirely to spaces and pairs having the homotopy type of CW complexes and CW pairs. The theorems of [17] and Proposition (0) of [21] are sufficient to ensure that none of our constructions leads outside this class.

2.1 We consider Eilenberg-MacLane spaces $K(G,n)$ satisfying the following conditions:

(i) $K(G,n)$ is a topological abelian group

(ii) the discrete automorphism group aut G acts on $K(G,n)$ by continuous group automorphisms (such that the induced action on $\pi_n K(G,n) \approx G$ is the obvious one).

Such $K(G,n)$ are easily constructed for any countable abelian group G and $n \geq 0$: one either takes the Milnor realisation [16] of the standard semi-simplicial abelian group $K(G,n)$ [5], or one observes that the construction of Milgram ([14],§4) yields such a space.

Let B be a connected space with base-point 0, and let $\mathcal{G}$ be a system of local coefficients on B with vertex group G (i.e. G is the group assigned to the base-point by $\mathcal{G}$). According to

Steenrod ([22], §31.1), we may regard $\mathcal{G}$ as a bundle over B with fibre G and group (essentially) aut G. Such a bundle is determined by a homomorphism $\pi_1 B \to$ aut G: that is, by a $\pi_1 B$-module structure on G. Taking K(G,n) as above, we form from $\mathcal{G}$ the associated bundle with fibre K(G,n). We denote this by $\epsilon : K_B(G,n) \to B$, and call it the <u>Eilenberg-MacLane object over B of type (G,n)</u> corresponding to the $\pi_1 B$-module G.

Since aut G acts by group automorphisms on K(G,n), $K_B(G,n) \to B$ is an abelian group object in the category of spaces over B. The zero of this structure is the section $B \to K_B(G,n)$ which picks out the zero in each fibre. We shall identify B with its image under this section.

The abelian group structure on K(G,n) induces group structures on the path space PK(G,n) and loop space $\Omega K(G,n)$, and aut G acts on these by automorphisms. Let $P_B K_B(G,n) \to B$ be the bundle associated to $\mathcal{G}$ with fibre PK(G,n). Then the end-point map $PK(G,n) \to K(G,n)$ yields a map of bundles $P_B K_B(G,n) \to K_B(G,n)$ which is a fibration of the total spaces. Restricting this fibration to $B \subset K_B(G,n)$, we obtain the bundle associated to $\mathcal{G}$ with fibre $\Omega K(G,n)$. This is an Eilenberg-MacLane object over B of type (G,n-1). Thus we have a diagram

$$\begin{array}{ccccc} K_B(G,n-1) = \Omega_B K_B(G,n) & \subset & P_B K_B(G,n) \\ \downarrow & & \downarrow \\ B & \subset & K_B(G,n) \end{array}$$

2.2 Let (X,A) be a pair and $f : X \to B$ be a map. Then we may take the induced bundle $f^*\mathcal{G}$ over X and form the cohomology groups $H^n(X,A;\ f^*\mathcal{G})$ of ([22],§31) (first replacing (X,A) by a homotopy equivalent CW pair if necessary).

We consider maps $g : (X,A) \to (K_B(G,n),B)$ which are fibrewise over f; that is, which satisfy $\epsilon g = f : X \to B$. We denote by $[(X,A),\ (K_B(G,n),B)]_f$ the set of equivalence classes of such maps under the relation of homotopy through maps of the same kind. The abelian group structure on $\epsilon\colon K_B(G,n) \to B$ induces a group structure on $[(X,A),(K_B(G,n),B)]_f$.

2.3 Proposition. $[(X,A),(K_B(G,n),B)]_f \approx H^n(X,A;f^*\mathcal{G})$.

Proof. It suffices to prove this when (X,A) is a CW pair. Consider the induced bundle $f^*\epsilon$ over X :

$$\begin{array}{ccc} E & \rightarrow & K_B(G,n) \\ {\scriptstyle f^*\epsilon}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ X & \xrightarrow{f} & B \end{array}$$

$f^*\epsilon$ has a zero section $s : X \to E$ induced from the zero section of ϵ. If h is any section of $f^*\epsilon$ such that $h|A = s|A$, we can from the primary difference $d(h,s) \in H^n(X,A;\ \mathcal{B}(\pi_n))$ as in [22], where $\mathcal{B}(\pi_n)$ denotes the 'bundle of n'th homotopy groups of the fibre' of $f^*\epsilon$. Clearly $\mathcal{B}(\pi_n) \approx f^*\mathcal{G}$. Consider the correspondence $\{h\} \to d(h,s)$. By Theorem 37.2 of [22], any element of $H^n(X,A;f^*\mathcal{G})$ can be realised as $d(h,s)$ for some section h over $X^{n+1} \cup A$: but then h extends to the whole of X because of the vanishing of the higher homotopy groups of the fibre. Likewise two sections give the same difference class iff they are homotopic over $X^n \cup A$: but then they are homotopic over X. Hence $\{h\} \to d(h,s)$ is a bijection. But there is an evident bijection between the set of homotopy classes $\{h\}$ and the set $[(X,A),(K_B(G,n),B)]_f$. The proposition follows.

In future we shall usually denote $H^n(X,A;f^*\mathcal{G})$ by $H^n(X,A;G)$ and call it the n'th cohomology group with coefficients in the $\pi_1 B$-module G. This only makes sense when a map $f : X \to B$ is specified, but it will be clear from the context which map is meant. 2.3 shows that $(K_B(G,n),B)$ represents the functor $H^n(\ ,\ ;G)$ on the category of pairs over B: hence there is a <u>universal class</u> $\hat{\iota}_n \in H^n(K_B(G,n),B;G)$.

2.4 <u>The Leray-Serre spectral sequence.</u>

Let $p : Y \to X$ be a fibration with connected fibre and base. Denote by F_x the fibre over $x \in X$. Suppose there is a local system of groups $\mathcal{G}:x \to \mathcal{G}_x$ on X with vertex group G. Then $x \to H^s(F_x;\mathcal{G}_x)$ determines a local system on X: for any path from x to $x' \in X$ induces a homotopy equivalence $F_{x'} \simeq F_x$ and an isomorphism $\mathcal{G}_x \approx \mathcal{G}_{x'}$, and hence an isomorphism $H^s(F_x;\mathcal{G}_x) \approx H^s(F_{x'};\mathcal{G}_{x'})$. Call this local system $H^s(F;\mathcal{G})$. Then by modifying standard methods one obtains a Leray-Serre spectral sequence

$$E_2^{r,s} = H^r(X;H^s(F,\mathcal{G})) \Rightarrow H^{r+s}(Y;p^*\mathcal{G}).$$

(The method of [4] adapts well if X has any kind of simplicial structure, and we can reduce the general case to this special one by replacing X by the realisation of its singular complex). The $E_2^{0,s}$ term is $H^0(X;H^s(F,\mathcal{G}))$, which is isomorphic to the invariant subgroup of $H^s(F;G)$ under the action of $\pi_1 X$ given by the local system $H^s(F;\mathcal{G})$. The transgression $\tau:E_2^{0,s} \to E_2^{s+1,0}$ is the additive relation $(p^*)^{-1}\delta$ in

$$H^o(X;H^s(F;\mathcal{G})) \subset H^s(F;G) \xrightarrow{\delta} H^{s+1}(Y,F;p^*\mathcal{G}) \xleftarrow{p^*} H^{s+1}(X,o;\mathcal{G}) \sim E_2^{s+1,o}$$

and the image of τ is the kernel of $p^*:H^{s+1}(X;\mathcal{G}) \to H^{s+1}(Y;p^*\mathcal{G})$.

2.5 <u>Lemma.</u> Let G be a $\pi_1 B$-module. Consider the spectral sequence of the fibration $\Omega K(G,n) \subset P_B K_B(G,n) \to K_B(G,n)$ with the natural local system on $K_B(G,n)$. Let $\iota_{n-1} \in H^{n-1}(\Omega K(G,n);G)$ and $\hat{\iota}_n \in H^n(K_B(G,n),B;G)$ be the universal classes. Then $\iota_{n-1} \in E_2^{o,n-1}$ and ι_{n-1} transgresses to $-\hat{\iota}_n$.

Proof. ι_{n-1} corresponds to the identity map in $\mathrm{Hom}(\pi_{n-1}\Omega K(G,n),G) \approx \mathrm{Hom}\,(G,G)$. Hence it is invariant under the action of $\pi_1 B$, and appears in $E_2^{o,n-1}$. Since $E^{i,j} = 0$ for $0 < j < n-1$, ι_{n-1} is transgressive. It transgresses to some element of $H^n(K_B(G,n),B;G) \subset H^n(K_B(G,n);G)$ (inclusion since B is a retract) because $H^n(B;G)$ maps isomorphically to $H^n(P_B K_B(G,n);G)$. Let $\widetilde{B}$ be the universal cover of B. Consider the diagram

$$\begin{array}{ccc}
\Omega K(G,n) & = & \Omega K(G,n) \\
\downarrow & & \downarrow \\
\widetilde{B} \times PK(G,n) & \to & P_B K_B(G,n) \\
\downarrow & & \downarrow \\
\widetilde{B} \times K(G,n) & \to & K_B(G,n)
\end{array}$$

obtained by pulling back the fibration to the universal cover $\widetilde{B} \times K(G,n)$ of the base. The fibration on the left is the product with $\widetilde{B}$ of the usual path fibration over $K(G,n)$, and the induced coefficients on $\widetilde{B} \times K(G,n)$ are untwisted: thus ι_{n-1} transgresses to $-(1 \times \iota_n) \in H^n(\widetilde{B} \times K(G,n), \widetilde{B}; G)$. The result follows by naturality of the transgression, since

$H^n(K_B(G,n),B;G) \to H^n(\widetilde{B} \times K(G,n),\widetilde{B};G)$ is a monomorphism, as follows from the spectral sequence of the fibration

$$\widetilde{B} \times K(G,n) \to K_B(G,n) \to K(\pi_1 B,1) .$$

2.6 <u>Notation</u>. In discussing k-invariants, we shall have to deal with elements of groups like $H^{n_1}(X,A;G_1) \oplus \dots \oplus H^{n_k}(X,A;G_k)$ where $G_1, \dots , G_k$ are $\pi_1 B$-modules and the coefficient bundles over X are all induced by a single fixed map $X \to B$. It seems worth while to have a systematic notation. We get this by introducing graded coefficient modules.

<u>Definition</u>. Let $\Gamma = \{\Gamma_i\}_{i \in Z}$ be a graded $\pi_1 B$-module, and let a map $X \to B$ be specified. We define the <u>generalised cohomology group</u> $\mathcal{H}^n(X,A;\Gamma)$ to be the direct product $\Pi_{i \in Z} H^{i+n}(X,A;\Gamma_i)$: we call its elements <u>generalised cohomology classes.</u>

Thus we deal with $H^{n_1}(X,A;G_1) \oplus \dots \oplus H^{n_k}(X,A;G_k)$ by setting $\Gamma_{n_i} = G_i$ for $i = 1, \dots, k$ and $\Gamma_j = 0$ otherwise: then the above group is $\mathcal{H}^0(X,A;\Gamma)$ and $H^{n_1-1}(X,A;G_1) \oplus \dots \oplus H^{n_k-1}(X,A;G_k)$ appears as $\mathcal{H}^{-1}(X,A;\Gamma)$.

The functor $H^{i+n}(\ ,\ ;\Gamma_i)$, defined on the category of pairs over B, is represented by the Eilenberg-MacLane fibration $K_B(\Gamma_i, n+i) \to B$. Hence $\mathcal{H}^n(\ ,\ ;\Gamma)$ is represented by the fibre product of these fibrations: we denote this <u>generalised Eilenberg-MacLane object</u> by $\mathcal{K}(\Gamma,n) \to B$. Taking the fibre product of the diagrams in 2.1, we obtain

$$\begin{array}{ccccc} \mathcal{K}(\Gamma,n-1) = \Omega_B\mathcal{K}(\Gamma,n) & \subset & P_B\mathcal{K}(\Gamma,n) \\ \downarrow & & \downarrow \\ B & \subset & \mathcal{K}(\Gamma,n) . \end{array}$$

We call $P_B\mathcal{K}(\Gamma,n) \to \mathcal{K}(\Gamma,n)$ the <u>universal fibration</u> over $\mathcal{K}(\Gamma,n)$.

We can obviously generalise the spectral sequence to the case of graded coefficients by taking direct products.

2.7 <u>Proposition</u>. Suppose we have a fibration $p : Y \to X$ with connected fibre F, a graded π_1B-module Γ, and a map $\phi: X \to B$. Let $\alpha \in \mathcal{H}^n(F;\Gamma)$ be a generalised class which is transgressive in the spectral sequence of p with coefficients Γ. Then there is a commutative diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \Pi_i \Omega K(\Gamma_i, n+i+1) \arrow[d] \\
Y \arrow[r] \arrow[d, "p"'] & P_B \mathcal{K}(\Gamma, n+1) \arrow[d] \\
X \arrow[r] \arrow[dr, "\phi"'] & \mathcal{K}(\Gamma, n+1) \arrow[d] \\
& B
\end{tikzcd}$$

such that $F \to \Pi_i \Omega K(\Gamma_i, n+i+1)$ represents α.

Proof. First take the special case when Γ is concentrated in one dimension m, i.e. $\Gamma_i = 0$ for $i \neq m$. Then $\alpha \in H^{n+m}(F;\Gamma_m)$. Let α transgress to $\beta \in H^{n+m+1}(X;\Gamma_m)$. By 2.3, the class $-\beta$ can be represented by a map $f : X \to K_B(\Gamma_m, n+m+1)$ which is fibrewise over B. Since $p^*\beta = 0$ there is a homotopy, fibrewise over B, from the composite $fp : Y \to K_B(\Gamma_m, n+m+1)$ to the zero map $Y \to B \subset K_B(\Gamma_m, n+m+1)$. The adjoint of such a homotopy is a map $\psi : Y \to P_B K_B(\Gamma_m, n+m+1)$ lifting fp. Consider $\psi|F : F \to \Omega K(\Gamma_m, n+m+1)$. This represents some class $\alpha' \in H^{n+m}(F;\Gamma_m)$ which, by 2.5 and naturality, transgresses to β. Hence $\alpha - \alpha'$ transgresses to zero, and so $\delta(\alpha - \alpha') \in H^{n+m+1}(Y,F;\Gamma_m)$ is zero. Therefore $\alpha - \alpha'$ is the restriction of some class $\gamma \in H^{n+m}(Y;\Gamma_m)$. Represent γ by $\psi' : Y \to \Omega_B K_B(\Gamma_m, n+m+1)$, and form $\psi + \psi' : Y \to P_B K_B(\Gamma_m, n+m+1)$ using the group structure of the object $P_B K_B(\Gamma_m, n+m+1) \to B$. Then $(\psi + \psi')|F$ represents the class $\alpha' + (\alpha - \alpha') = \alpha$, and $\psi + \psi'$ still lifts $f : X \to K_B(\Gamma_m, n+m+1)$. This establishes the special case. We deduce the general result by treating each Γ_i separately, and forming fibre products over B.

2.8 Definition. Let $p : Y \to X$ be a fibration with connected fibre F, and $\phi : X \to B$ a map. We say $(Y \xrightarrow{q} Y_1 \xrightarrow{p_1} X, \Gamma, k)$ is an elementary factorisation of p over ϕ if

(i) Γ is a graded $\pi_1 B$-module and $k \in \mathcal{H}^1(X;\Gamma)$

(ii) $p_1 : Y_1 \to X$ is induced from the universal fibration by $k : X \to \mathcal{K}(\Gamma,1)$, and $p_1 q = p$: hence there is a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "q"] \arrow[dr, "p"'] & Y_1 \arrow[rr] \arrow[d, "p_1"] & & P_B\mathcal{K}(\Gamma,1) \arrow[d] \\
& X \arrow[rr, "k"] \arrow[dr, "\phi"'] & & \mathcal{K}(\Gamma,1) \arrow[dl, dashed] \\
& & B &
\end{tikzcd}$$

(iii) $q|F : F \to \prod_i \Omega K(\Gamma_i, i+1)$ induces an epimorphism of homotopy groups in every dimension (i.e. $q|F$ represents a spherical set of cohomology classes in the sense of [13], [23]).

We refer to Γ as the <u>coefficient module</u>, and to k as the <u>k-invariant</u>, of the factorisation.

Let F_1 be the fibre of $q : Y \to Y_1$. Then F_1 is homotopy equivalent to the fibre of $q|F : F \to \prod_i \Omega K(\Gamma_i, i+1)$, so by (iii) there are short exact sequences

$$0 \to \pi_i F_1 \to \pi_i F \to \Gamma_i \to 0 .$$

2.9 <u>Definition.</u> Let $p : Y \to B$ be a fibration with fibre F, where F, Y, B are connected. A <u>modified Postnikov tower (MPT) $\mathcal{M}$ for p</u> (with local coefficients) of height s is a diagram

$$\begin{array}{cccccccccccc}
 & & & & \mathcal{K}(\Gamma^{s-1},1) & & & \mathcal{K}(\Gamma^{1},1) & & \mathcal{K}(\Gamma^{o},1) & & \\
 & & & & k^{s-1}\uparrow & & & k^{1}\uparrow & & k^{o}\uparrow & & \\
\mathcal{M} : Y & \to & Y^{s} & \to & Y^{s-1} & \to & \cdots \to & Y^{1} & \to & Y^{o} & = & B
\end{array}$$

such that (i) the composite $Y \to Y^o$ is p

(ii) for $0 \leq t < s$, $(Y \to Y^{t+1} \to Y^t, \Gamma^t, k^t)$ is an elementary factorisation of $Y \to Y^t$ over $Y^t \to B$.

Let F^t be the fibre of $Y \to Y^t$. Then $F^o = F$ and there are short exact sequences $0 \to \pi_i F^{t+1} \to \pi_i F^t \to \Gamma_i^t \to 0$ for $0 \leqslant t < s$, all i.

We say $\mathcal{M}$ kills homotopy groups up to $\pi_r F$ if F^s is r-connected: that is, if $Y \to Y^s$ is an (r+1)-equivalence. At the other extreme, $\mathcal{M}$ leaves $\pi_r F$ unaltered if $\pi_r F^s \to \pi_r F$ is iso : that is, if $\Gamma^t_r = 0$ for each t.

The MPT $\mathcal{M}$ is more useful for computations if the coefficient modules Γ^t are restricted. We say $\mathcal{M}$ has sensible coefficients if the underlying abelian group of each Γ^t_i is finitely-generated and is a direct sum of a free abelian group with vector spaces over Z_p for various primes p.

We shall sometimes wish to alter the grading in each Γ^t by some integer n, so that $k^t \in \mathcal{H}^{n+1}(Y^t; \Gamma^t)$. This evidently causes no trouble.

2.10 The following existence theorem, based on 2.7, constructs a tower of unnecessarily large height. It is not really an analogue of 2.2.5 of [13]. We shall construct better towers in practical cases by applying 2.7 directly.

<u>Theorem.</u> Let $p: Y \to B$ be a fibration with simply-connected fibre F and connected base B. Suppose $\pi_2 F, \ldots, \pi_r F$ are finitely generated. Then

(i) p has an MPT $\mathcal{M}$ with sensible coefficients which kills homotopy groups up to $\pi_r F$ and leaves $\pi_s F$ unaltered for all $s > r$.

(ii) if an MPT $\mathcal{M}'$ for p is given which has sensible coefficients and leaves $\pi_s F$ unaltered for $s > r$, then $\mathcal{M}'$ can be extended (by the addition of extra stages) to $\mathcal{M}$ satisfying (i).

Proof. It suffices to prove (ii), because we can always deduce (i) by taking $\mathcal{M}'$ to be the trivial tower of height 0. So suppose $\mathcal{M}'$ is the tower

$$\begin{tikzcd}
 & & \mathcal{K}(\Gamma^{n-1},1) & & \mathcal{K}(\Gamma^{o},1) \\
Y \arrow[r] & Y^n \arrow[r] & Y^{n-1} \arrow[r] \arrow[u, "k^{n-1}"] & \cdots \arrow[r] & Y^o = B \arrow[u, "k^o"]
\end{tikzcd}$$

Suppose the fibre F^n of $Y \to Y^n$ is exactly $(q-1)$-connected. We work by induction on q: specifically, we show that we can add finitely many stages $Y^{n+1}, \ldots, Y^{n+m}$ to the tower such that the fibre of $Y \to Y^{n+m}$ is q-connected. Then, provided $\Gamma^n, \ldots, \Gamma^{n+m-1}$ are sensible and have zero components in gradings $> r$, the result will follow by induction.

We construct Y^{n+1} in such a way as to get rid of the free abelian part of $\pi_q F^n$. F^n is certainly 1-connected, so the fibration $F^n \to Y \to Y^n$ determines a natural action of $\pi_1 Y^n$ on $\pi_q F^n$. This action preserves the torsion subgroup $T \subset \pi_q F^n$, so $\pi_1 Y^n$ acts on the quotient $Q = \pi_q F^n/T$, which is free abelian of finite rank. But $\pi_1 Y^n \approx \pi_1 B$, since the original fibre F was 1-connected. Thus Q is a $\pi_1 B$-module. Consider the spectral sequence of $F^n \to Y \to Y^n$ with coefficients in this module. Since F^n is $(q-1)$-connected, $H^q(F^n;Q) \approx \mathrm{Hom}_Z(\pi_q F^n, Q)$. Let $\alpha \in H^q(F^n;Q)$ correspond to the projection $\pi_q F^n \to \pi_q F^n/T = Q$. Then α is invariant under the action of $\pi_1 B$, so $\alpha \in E_2^{o,q} \approx H^o(Y^n;H^q(F^n;Q))$; and α is transgressive since F^n is $(q-1)$-connected. Thus by 2.7 we can construct

$$\begin{tikzcd}
F^n \arrow[r] \arrow[d] & \Omega K(Q,q+1) \arrow[d] \\
Y \arrow[r] \arrow[d] & P_B\mathcal{K}(\Gamma^n,1) \arrow[d] \\
Y^n \arrow[r, "k^n"] \arrow[dr] & \mathcal{K}(\Gamma^n,1) \arrow[d] \\
& B
\end{tikzcd}$$

(where $\Gamma_q^n = Q$, $\Gamma_i^n = 0$ for $i \neq q$) such that $F^n \to \Omega K(Q,q+1)$ induces $\pi_q F^n \to Q$. Let $Y^{n+1} \to Y^n$ be the fibration induced by k^n: then $Y \to Y^n$ has an elementary factorisation $(Y \to Y^{n+1} \to Y^n, \Gamma^n, k^n)$. The short exact sequences of 2.8 show that the fibre F^{n+1} of $Y \to Y^{n+1}$ satisfies $\pi_i F^{n+1} \approx \pi_i F^n$ for $i \neq q$, while $\pi_q F^{n+1}$ is the (finite) torsion subgroup $T \subset \pi_q F^n$.

If $T = 0$, the inductive step is complete: otherwise write T as the direct sum of non-trivial primary components $T_1 \oplus \dots \oplus T_\mu$ corresponding to distinct primes $p_1, \dots p_\mu$. Let p_1^N be the highest power of p_1 which occurs as the order of an element in T_1, and let S be the subgroup of T generated by $T_2 \oplus \dots \oplus T_\mu \cup \{x \in T_1 | p_1^{N-1}x = 0\}$. Then $V = T/S$ is a non-trivial vector space over Z_{p_1}. The action of $\pi_1 Y^{n+1} \approx \pi_1 B$ on T determined by the fibration $F^{n+1} \to Y \to Y^{n+1}$ preserves the subgroup S, so gives an action of $\pi_1 B$ on V. Let $\beta \in H^q(F^{n+1};V)$ be the element corresponding to the projection $\pi_q F^{n+1} = T \to V$; then β is invariant under $\pi_1 B$, so appears (and is transgressive) in the spectral sequence of $F^{n+1} \to Y \to Y^{n+1}$ with coefficients in the $\pi_1 B$-module V. Applying 2.7 again we obtain an elementary factorisation $Y \to Y^{n+2} \to Y^{n+1}$ such that $\pi_q F^{n+2} \approx S$ and $\pi_i F^{n+2} \approx \pi_i F^{n+1}$ for $i \neq q$. But $|S| < |T|$, so by induction we can add finitely many stages $Y^{n+2}, \dots, Y^{n+m}$ to the tower until the fibre of $Y \to Y^{n+m}$ is q-connected. This completes the induction.

2.11. We construct similarly the canonical Postnikov tower (of height n) of any fibration $p:Y \to B$ with 1-connected fibre F. This is an MPT

$$\begin{array}{ccccccccc} & & & \mathcal{K}(\Gamma^{n-1},1) & & \mathcal{K}(\Gamma^{o},1) & \\ & & & k^{n-1}\uparrow & & k^{o}\uparrow & \\ Y & \to & Y^n & \to & Y^{n-1} & \to \dots \to & Y^o = B \end{array}$$

in which the homotopy groups of the fibre F^t of $Y \to Y^t$ satisfy $\pi_i F^t = 0$ for $i < t$ and $\pi_i F^t \approx \pi_i F$ for $i \geq t$. The k-invariant k^t is the transgression of the fundamental class of $H^t(F^t;\pi_t F)$ in the spectral sequence of $F^t \to Y \to Y^t$ with coefficients corresponding to the natural $\pi_1 B$-structure on $\pi_t F$. This is just the local-coefficient version of the method of Hermann [7].

§3. Stable modified Postnikov towers.

3.0 We seek a way of killing the stable homotopy groups of the fibre of $F \subset X \xrightarrow{p} B$. We shall take an MPT for p which kills those homotopy groups $\pi_i F$ which happen to lie in the stable range $i \leq 2 \operatorname{conn} F$. We obtain from this, by 3.3 below, an MPT for σp which kills the corresponding homotopy groups of the fibre ΣF. We extend this tower to kill the next homotopy group, which is now in the stable range, then proceed to $\sigma^2 p$, and so on. The resulting 'stable tower' is the raw material for the spectral sequence of §4.

3.1 We have to deal with many fibrations with base B. We adopt the following abuse of notation: if $p: X \to B$ is a fibration, then σX denotes the total space of σp (1.2).

We introduce a <u>suspension homomorphism</u> $S : \mathcal{H}^n(X;\Gamma) \to \mathcal{H}^{n+1}(\sigma X, B;\Gamma)$ where B is embedded in σX by the canonical section s_0 of σp, and Γ is a graded $\pi_1 B$-module. Define $\sigma_+ X = \{(x,t) \in \sigma X \mid t \geq \frac{1}{2}\}$, $\sigma_- X = \{(x,t) \in \sigma X \mid t \leq \frac{1}{2}\}$. Then $\sigma_+ X \cup \sigma_- X = \sigma X$ and $\sigma_+ X \cap \sigma_- X$ is homeomorphic to X. $\sigma_+ X$, $\sigma_- X$ are copies of the mapping cylinder of p, apart from a change in topology which, as in 1.2, does not affect homotopy type. We define S to be the composite

$$\mathcal{H}^n(X;\Gamma) \xrightarrow{\delta} \mathcal{H}^{n+1}(\sigma_+X,X;\Gamma) \approx \mathcal{H}^{n+1}(\sigma X,\sigma_-X;\Gamma) \approx \mathcal{H}^{n+1}(\sigma X,B;\Gamma)$$

where the first isomorphism arises from excision and the second from the homotopy equivalence $B \subset \sigma_-X$. If p happens to admit a section $s:B \subset X$, then the composite

$$\mathcal{H}^n(X,B;\Gamma) \to \mathcal{H}^n(X;\Gamma) \xrightarrow{S} \mathcal{H}^{n+1}(\sigma X,B;\Gamma)$$

is an isomorphism. When B is a point, this reduces to the usual suspension isomorphism.

3.2 Let $p : X \to B$ be a fibration with connected fibre F. Suppose we have an MPT

$$\begin{array}{lccccccc} & & & \mathcal{K}(\Gamma^{n-1},1) & & & \mathcal{K}(\Gamma^{o},1) & \\ & & & k^{n-1}\uparrow & & & k^{o}\uparrow & \\ \mathcal{M}: & X & \to X^n \to & X^{n-1} & \to & \dots & \to X^o & = B \end{array}$$

for p. We apply the functor σ to each fibration $X^i \to B$, and obtain a diagram

$$\sigma X \to \sigma X^n \to \sigma X^{n-1} \to \dots \to \sigma X^o \simeq B.$$

Each $\sigma X^i \to B$ has a canonical section s_o, and $\sigma X^{i+1} \to \sigma X^i$ is section-preserving. After 3.1, we have the suspended k-invariants $Sk^i \in \mathcal{H}^2(\sigma X^i,B;\Gamma^i) \subset \mathcal{H}^2(\sigma X^i,\Gamma^i)$.

3.3 <u>Theorem.</u> Suppose we have the data of 3.2, where F has connectivity $c \geqslant 1$ and $\mathcal{M}$ kills homotopy groups up to $\pi_{2c}F$ but leaves $\pi_j F$ unaltered for $j > 2c$. Then

(i) the fibration σp has a unique MPT $\sigma\mathcal{M}$ which fits as the top ~~now~~ row in a commutative diagram

$$\begin{tikzcd}
 & & \mathcal{K}(\Gamma^{n-1},2) & & \mathcal{K}(\Gamma^{0},2) & \\
 & Y^n \arrow[r] & Y^{n-1} \arrow[u, "\kappa^{n-1}"] \arrow[r] & \cdots \arrow[r] & Y^0 \arrow[u, "\kappa^0"'] \arrow[dr, equal] & \\
\sigma X \arrow[ur] \arrow[dr] & & & & & B \\
 & \sigma X^n \arrow[r] \arrow[uu, "f_n"] & \sigma X^{n-1} \arrow[r] \arrow[uu, "f_{n-1}"] & \cdots \arrow[r] & \sigma X^0 \arrow[uu, "f_0"'] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

and whose k-invariants κ^i satisfy $f_i^* \kappa^i = -Sk^i$.

(ii) This $\sigma\mathcal{M}$ kills homotopy groups up to $\pi_{2c+1}(\Sigma F)$ and leaves $\pi_j(\Sigma F)$ unaltered for $j > 2c + 1$.

(iii) In the above diagram, each map f_i is a $(2c+3)$-equivalence.

A proof of 3.3 is given in Appendix 1. We note that the k-invariant κ^r of $\sigma\mathcal{M}$ is located in $\mathcal{H}^2(Y^r;\Gamma^r)$. We choose not to alter the grading in Γ^r to bring it into $\mathcal{H}^1$.

3.4 <u>Definition</u>. Let $F \subset X \xrightarrow{p} B$ be a fibration with B connected and F c-connected, $c \geq 1$. A <u>stable modified Postnikov tower</u> (SMPT) for p is a sequence $\{\mathcal{M}_i\}_{i \geq 0}$ such that

(i) $\mathcal{M}_i$ is an MPT for $\sigma^i p$ (where $\sigma^0 p = p$)

(ii) $\mathcal{M}_i$ kills homotopy groups up to $\pi_{2c+2i}(\Sigma^i F)$ and leaves $\pi_j(\Sigma^i F)$ unaltered for $j > 2c + 2i$.

(iii) $\mathcal{M}_{i+1}$ is an extension of $\sigma\mathcal{M}_i$ (i.e. is obtained from the tower $\sigma\mathcal{M}_i$ defined in 3.3 by the addition of extra stages).

3.5 Let $\mathcal{M}_{i+1}$ be the tower $\sigma^{i+1}X \to Z^n \to \ldots \to Z^0 = B$. By 3.4 (iii) and 3.3 there is a diagram

$$\begin{array}{ccccccccccc} & \nearrow & Z^n & \to \ldots \to & Z^{m+1} & \to & Z^m & \to & Z^{m-1} & \to \ldots \to & Z^0 & \searrow^{=} & \\ \sigma^{i+1}X & & & & & & \uparrow & & \uparrow & & \uparrow & & B \\ & \searrow & & \longrightarrow & & & \sigma Y^m & \to & \sigma Y^{m-1} & \to \ldots \to & \sigma Y^0 & \nearrow_{\simeq} & \end{array}$$

which is to be considered part of the structure of the SMPT.

Convention. Let Γ^r be the (graded) coefficient module at the r'th stage of $\mathcal{M}_i$, and k^r the r'th k-invariant. Then we choose the grading in Γ^r so that $k^r \in \mathcal{H}^{i+1}(Y^r;\Gamma^r)$ instead of $\mathcal{H}^1$. This ensures that the sequence $\{\Gamma^0,\Gamma^1, \dots , \Gamma^n\}$ of coefficient modules of $\mathcal{M}_{i+1}$ is an extension of the sequence of coefficient modules of $\mathcal{M}_i$. Thus we can speak of the sequence $\{\Gamma^0,\Gamma^1, \dots \}$ of coefficient modules of the SMPT.

3.6 Proposition. Let p be a fibration as in 3.4 and let $\mathcal{M}$ be any MPT for p which does not alter $\pi_j F$ for $j > 2c$. Then there is an SMPT $\{\mathcal{M}_i\}_{i\geq 0}$ for p with $\mathcal{M}_0$ an extension of $\mathcal{M}$. Each $\mathcal{M}_i$ can be chosen to have sensible coefficients if $\mathcal{M}$ has.

Proof. We obtain $\mathcal{M}_0$ by extending $\mathcal{M}$ to kill homotopy groups up to $\pi_{2c}F$ by 2.10, using sensible coefficients if required. Then we work by induction: given $\mathcal{M}_{i-1}$, we construct $\sigma\mathcal{M}_{i-1}$ by 3.3 and extend it to kill $\pi_{2c+2i}(\Sigma^i F)$ by 2.10. This gives $\mathcal{M}_i$.

3.7 Using 2.11 instead of 2.10, we construct the canonical stable Postnikov tower of p, whose r'th coefficient module Γ^r has the stable homotopy group $\pi_r^\Sigma F$ in degree r and zero in other degrees.

§4 A spectral sequence for $\pi_*^\sigma(p)$

4.0 We suppose given a fibration $p:X \to B$ whose base is a connected CW complex and whose fibre F is c-connected, $c \geq 1$. From an SMPT $\{\mathcal{M}_i\}_{i\geq 0}$ with coefficient modules Γ^0, Γ^1, ..., we obtain a spectral sequence with $E_1^{s,t} = \mathcal{H}^{t-s}(B;\Gamma^s)$ and abutment $\pi_*^\sigma(p)$. The spectral sequence arises from an $H(p,q)$-system ([2], [3]) which is the direct limit of a sequence of approximations obtained from the towers $\mathcal{M}_i$.

4.1 Lemma Let

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

be a commutative diagram of fibrations, where B is a CW complex. Then the induced map $\sec p_1 \to \sec p_2$ has the homotopy lifting property for complexes (i.e. is a Serre fibration, but need not be surjective).

Proof Let $g_t : A \to \sec p_2$ be a homotopy of a complex A in $\sec p_2$. Since $\sec p_2 \subset X_2^B$, g_t is adjoint to a homotopy $\hat{g}_t : A \times B \to X_2$. Since f is a fibration, $\hat{g}_t$ lifts to a homotopy of $A \times B$ in X_1 with any given initial position. Taking the adjoint, we obtain a lift of g_t to $\sec p_1$ with any given initial position.

4.2 Let $p: X \to B$ be a fibration with c-connected fibre F and $\{\mathcal{M}_i\}_{i \geq 0}$ an SMPT for p, as in 4.0. Let h_i be the height of the tower $\mathcal{M}_i$. We construct a diagram from $\mathcal{M}_i$, $i \geq 1$, using the fact that each stage in the tower has a canonical section (over B) derived from the canonical section s_0 of $\sigma^i p$. If a section for $\sigma^0 p = p$ is specified, the construction also applies to the tower $\mathcal{M}_0$.

Let $\mathcal{M}_i$ be

$$\begin{array}{ccccccccc} & & \mathcal{K}(\Gamma^{h_i - 1}, i+1) & & & & \mathcal{K}(\Gamma^{0}, i+1) & & \\ & & {\scriptstyle k^{h_i-1}}\uparrow & & & & {\scriptstyle k^{0}}\uparrow & & \\ \sigma^i X & \to & Y_0^{h_i} & \to & Y_0^{h_i - 1} & \to \ldots \to & Y_0^{0} & = & B. \end{array}$$

If we embed $B \subset Y_0^s$ by the canonical section, then the k-invariant $k^s \in \mathcal{H}^{i+1}(Y_0^s ; \Gamma^s)$ restricts to zero in $\mathcal{H}^{i+1}(B; \Gamma^s)$, because $B \subset Y_0^s$ lifts to $B \subset Y_0^{s+1}$. Thus k^s lies in the subgroup $\mathcal{H}^{i+1}(Y_0^s, B; \Gamma^s)$. Therefore we may adjust the tower so that each k^s is section-preserving, i.e. carries the canonical section of $Y_0^s \to B$ to the zero section of $\mathcal{K}(\Gamma^s, i+1)$.

We assume this done in what follows.

We construct the <u>triangular diagram</u> of $\mathcal{M}_i$:

$$
\begin{array}{lcccccccccccl}
B & = & Y^{\infty}_{\infty} & \subset & Y^{\infty}_{h_i} & \subset \cdots \subset & Y^{\infty}_{2} & \subset & Y^{\infty}_{1} & \subset & Y^{\infty}_{0} & \simeq & \sigma^i X \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \swarrow & \\
 & & B & = & Y^{h_i}_{h_i} & \subset \cdots \subset & Y^{h_i}_{2} & \subset & Y^{h_i}_{1} & \subset & Y^{h_i}_{0} & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & & \vdots & & \vdots & & \\
 & & & & B & = & Y^{s}_{s} \subset \cdots & \subset & Y^{s}_{1} & \subset & Y^{s}_{0} & \xrightarrow{k^s} & \mathcal{K}(\Gamma^s, i+1) \\
 & & & & & & & & \vdots & & \vdots & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & B & = & Y^{1}_{1} & \subset & Y^{1}_{0} & \xrightarrow{k^1} & \mathcal{K}(\Gamma^1, i+1) \\
 & & & & & & & & & & \downarrow & & \\
 & & & & & & & & B & = & Y^{0}_{0} & \rightarrow & \mathcal{K}(\Gamma^0, i+1)
\end{array}
\tag{4.3}
$$

The right-hand column is the MPT, and $Y^{\infty}_0 \to Y^{h_i}_0$ is a fibration homotopy equivalent to the map $\sigma^i X \to Y^{h_i}_0$. Each Y^s_s is B, embedded in Y_0^s by the section; and Y_r^s is the inverse image of $Y_r^r \subset Y_0^r$ under the fibration $Y_0^s \to Y_0^r$. Thus all the vertical maps are fibrations, and each commutative square is a pullback (induced fibration). Each Y_r^s is the total space of a sectioned fibration over B.

4.4 <u>Proposition</u> Y_s^{s+1} is a generalised Eilenberg-MacLane object over B of type $\mathcal{K}(\Gamma^s,i)$.

Proof. By the definition of MPT and the construction above we have induced fibrations

$$\begin{array}{ccccccc} & & Y_s^{s+1} & \subset & Y_0^{s+1} & \to & P_B\mathcal{K}(\Gamma^s,i+1) \\ & & \downarrow & & \downarrow & & \downarrow \\ B & = & Y_s^s & \subset & Y_0^s & \xrightarrow{k^s} & \mathcal{K}(\Gamma^s,i+1) \end{array}$$

The bottom row is the zero section of $\mathcal{K}(\Gamma^s,i+1)$, so by the diagram in 2.6 the induced fibration over B is $\mathcal{K}(\Gamma^s,i)$.

We now apply the functor sec to each of the fibrations over B in the triangular diagram. We write sec Y_r^s as an abbreviation for sec $(Y_r^s \to B)$.

4.5 <u>Proposition</u> Let $r,s,t \in \{0,1,\dots,h_i,\infty\}$, $r \leq s \leq t$. Then (i) the map $\sec Y_r^t \to \sec Y_r^s$ is a Serre fibration (not necessarily surjective): its fibre is $\sec Y_s^t$.

(ii) In the special case $t = s + 1$, the fibration sequence $\sec Y_s^{s+1} \to \sec Y_r^{s+1} \to \sec Y_r^s$ can be continued one stage to the right by the map $\sec Y_r^s \to \sec \mathcal{K}(\Gamma^s, i+1)$ induced by $Y_r^s \subset Y_0^s \xrightarrow{k^s} \mathcal{K}(\Gamma^s, i+1)$ (i.e. $\sec Y_r^{s+1}$ is the fibre of this map $\sec Y_r^s \to \sec \mathcal{K}(\Gamma^s, i+1)$).

Proof. (i) $\sec Y_r^t \to \sec Y_r^s$ is a Serre fibration by 4.1. The pullback square

$$\begin{array}{ccccc} & & Y_s^t & \subset & Y_r^t \\ & & \downarrow & & \downarrow \\ B & = & Y_s^s & \subset & Y_r^s \end{array}$$

shows that the inverse image of the canonical section is $\sec Y_s^t$.

(ii) We apply sec to the pullback diagram

$$\begin{array}{ccc} Y_r^{s+1} & \to & P_B\mathcal{K}(\Gamma^s, i+1) \\ \downarrow & & \downarrow \\ Y_r^s & \to & \mathcal{K}(\Gamma^s, i+1) \end{array} \quad \text{to obtain} \quad \begin{array}{ccc} \sec Y_r^{s+1} & \to & \sec[P_B\mathcal{K}(\Gamma^s, i+1)] \\ \downarrow & & \downarrow \\ \sec Y_r^s & \to & \sec \mathcal{K}(\Gamma^s, i+1) \end{array}$$

which is also a pullback. But $P_B\mathcal{K}(\Gamma^s,i+1)$ is homeomorphic to the space of paths in $\mathcal{K}(\Gamma^s,i+1)$ which begin in the zero section and lie entirely in some fibre of $\mathcal{K}(\Gamma^s,i+1) \to B$. Hence $\sec\,[P_B\mathcal{K}(\Gamma^s,i+1)]$ is homeomorphic to the path space of $\sec\mathcal{K}(\Gamma^s,i+1)$. Therefore the right-hand diagram above is exactly the diagram one uses to construct the fibre of the map $\sec Y_r^s \to \sec\mathcal{K}(\Gamma^s,i+1)$ by the mapping path construction.

4.6 By 4.5 (i) there are exact homotopy sequences

$$\cdots \to \pi_{n+1} \sec Y_r^s \xrightarrow{\partial} \pi_n \sec Y_s^t \to \pi_n \sec Y_r^t \to \cdots \to \pi_0 \sec Y_r^s$$

We should like to define an $M(p,q)$ - system ${}_iQ_*(\ ,\ )$ by

$${}_iQ_n(s,t) = \pi_{i+n} \sec Y_s^t$$

for $s,\ t \in \{0,1,\ldots,h_i,\infty\}$, $s \leq t$ and all $n \in Z$, with the convention that $\pi_j = 0$ for $j < 0$. (i is fixed, and indicates we are dealing with the tower $\mathcal{M}_i$). The structural maps ${}_iQ_n(r,s) \to {}_iQ_n(r',s')$ and boundaries $\partial : {}_iQ_n(r,s) \to {}_iQ_{n-1}(s,t)$ required are those in the homotopy sequences above. However, we do not know that $\pi_0 \sec Y_r^t \to \pi_0 \sec Y_r^s$ is epi. Therefore we have only a 'partially exact $H(p,q)$-system': we can only guarantee that the sequence of a triple (r,s,t) is exact as far as ${}_iQ_{-i}(r,s)$. We shall remedy this by taking a direct limit.

4.7 Let $\mathcal{M}_{i+1}$ be the tower $\sigma^{i+1}X \to Z_0^{h_{i+1}} \to \dots \to Z_0^o = B$. We form from this a triangular diagram containing spaces Z_s^t, and a partially exact $H(p,q)$-system ${}_{i+1}Q_n(s,t)$ as above, where $s,t \in \{0,1,\dots,h_{i+1},\infty\}$. We wish to compare the triangular diagram $\{Z_s^t\}$ with the triangular diagram $\{\sigma Y_s^t\}$ obtained by applying the functor σ to each fibration $Y_s^t \to B$ in diagram 4.3.
From 3.5 we have a commutative diagram

$$\begin{tikzcd}
 & Z_0^\infty \arrow[r] & Z_0^{h_{i+1}} \arrow[r] & \dots \arrow[r] & Z_0^{h_i} \arrow[r] & Z_0^{h_i-1} \arrow[r] & \dots \arrow[r] & Z_0^o \arrow[dr, equal] & \\
\sigma^{i+1}X \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & & & & & & & B \\
 & \sigma Y_0^\infty \arrow[uu, "\simeq", "f_\infty"'] \arrow[rrr] & & & \sigma Y_0^{h_i} \arrow[uu, "f_{h_i}"] \arrow[r] & \sigma Y_0^{h_i-1} \arrow[uu, "f_{h_i-1}"'] \arrow[r] & .. \arrow[r] & \sigma Y_0^o \arrow[uu, "f_o"] \arrow[ur] &
\end{tikzcd}$$

of sectioned fibrations over B. Each σY_0^s is filtered by $B \simeq \sigma Y_s^s \subset \sigma Y_{s-1}^s \subset \dots \subset \sigma Y_0^s$. We alter each $f_s : \sigma Y_0^s \to Z_0^s$ in turn by a homotopy, fibrewise over Z_0^{s-1}, to make f_s carry the subspace $\sigma Y_s^s \subset \sigma Y_0^s$ into $Z_s^s = B$. By using the homotopy lifting property we can ensure the above diagram remains commutative. Then f_t maps σY_s^t into $Z_s^t \subset Z_0^t$. We consider the composite

$${}_iQ_n(s,t) = \pi_{i+n}\sec Y_s^t \xrightarrow{\sigma_*} \pi_{i+1+n}\sec \sigma Y_s^t \xrightarrow{f_{t*}} \pi_{i+1+n}\sec Z_s^t = {}_{i+1}Q_n(s,t)$$

where σ_* is the homomorphism of 1.6. These composites commute with the structural morphisms and boundaries of the systems ${}_iQ_*(\ ,\ )$, ${}_{i+1}Q_*(\ ,\ )$, and so determine a morphism of partially exact $H(p,q)$-systems.

By iteration we obtain direct systems

$$\cdots \to {}_iQ_n(s,t) \to {}_{i+1}Q_n(s,t) \to {}_{i+2}Q_n(s,t) \to \cdots$$

4.8 <u>Definition</u> The <u>$H(p,q)$-system of the SMPT</u> $\{\mathcal{M}_i\}_{i\geq 0}$ has groups

$$Q_n(s,t) = \varinjlim_i {}_iQ_n(s,t)$$

for $n \in Z$ and $s,t \in \{0,1,\ldots,\infty\}$, $s \leq t$. Restriction maps $Q_n(s,t) \to Q_n(s',t')$ $(s' \leq s,\ t' \leq t)$ and boundaries $\partial: Q_n(r,s) \to Q_{n-1}(s,t)$ $(r \leq s \leq t)$ are defined to be the direct limits of the corresponding maps in the systems ${}_iQ_*(\ ,\ )$.

4.9 <u>Proposition</u> Q_* is an $H(p,q)$-system, and $Q_n(0,\infty) \approx \pi_n^\sigma(p)$.

Proof. We must verify the exactness axiom. For any triple $r \leq s \leq t$, the sequence in ${}_iQ_*$

$$\cdots \xrightarrow{\partial} {}_iQ_n(s,t) \to {}_iQ_n(r,t) \to {}_iQ_n(r,s) \xrightarrow{\partial} \cdots$$

is exact for $n \geq -i$. Thus the region of exactness increases as i increases. Since $\varinjlim$ is an exact functor, the corresponding sequence in Q_* is exact.

${}_iQ_n(0,\infty) = \pi_{i+n} \sec Y_0^\infty \approx \pi_{i+n} \sec \sigma^i p$. Under this identification the morphism ${}_iQ_n(0,\infty) \to {}_{i+1}Q_n(0,\infty)$ becomes $\sigma_*: \pi_{i+n} \sec \sigma^i p \to \pi_{i+1+n} \sec \sigma^{i+1} p$.

Hence $Q_n(0,\infty) \sim \varinjlim \pi_{n+i} \sec \sigma^i p = \pi_n^\sigma(p)$.

The $H(p,q)$-system Q_* yields a spectral sequence with abutment $Q_*(0,\infty) \approx \pi_*^\sigma(p)$ and $E_1^{s,t} = Q_{s-t}(s,s+1)$. With this notation the total degree is $s-t$, and $d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,t+r+1}$. We next identify $E_1^{s,t}$.

4.9 <u>Spaces of fibrewise loops</u> Suppose $q:Y \to B$ is a fibration with a given section $s:B \to Y$. We obtain a fibration $\Omega_B q:\Omega_B Y \to B$ as follows. $\Omega_B Y$ is the space of fibrewise loops $\{\omega:I \to Y \mid \omega(\partial I) \in s(B)\ ;\ q\omega(I) = q\omega(0)\}$, and $\Omega_B q$ maps the loop ω to the point $q\omega(I) \in B$. A map $q \to q'$ of sectioned fibrations over B induces a map $\Omega_B q \to \Omega_B q'$, so Ω_B becomes a functor. In the special case where $Y \to B$ is an Eilenberg-MacLane object $\mathcal{K}(\Gamma,n) \to B$, the fibration $\Omega_B\mathcal{K}(\Gamma,n) \to B$ is homeomorphic to the bundle also called $\Omega_B\mathcal{K}(\Gamma,n)$ in 2.6, and hence is a $\mathcal{K}(\Gamma,n-1)$.

There is an evident isomorphism of functors $\sec(\Omega_B q) \approx \Omega \sec q$ induced by the isomorphism $(Y^I)^B \approx (Y^B)^I$.

4.10 Proposition. The $H(p,q)$-system Q_* has the properties
(i) $E_1^{s,t} \approx \mathcal{H}^{t-s}(B;\Gamma^s)$.
(ii) ${}_iQ_n(s,t) \to Q_n(s,t)$ is an isomorphism if $i > -n$, $t \leqslant h_i$.
(iii) ${}_iQ_n(0,\infty) \to Q_n(0,\infty)$ is an isomorphism if $i \geqslant n + \dim B - 2c$ (where $c = \operatorname{conn} F$).

Proof. $E_1^{s,t} = Q_{s-t}(s,s+1)$. Choose i so large that $s - t + i \geqslant 0$ and $s + 1 \leqslant h_i$. Then ${}_iQ_{s-t}(s,s+1)$ is defined and

$$
\begin{aligned}
{}_iQ_{s-t}(s,s+1) &= \pi_{s-t+i}(\sec Y_s^{s+1}) \\
&\approx \pi_{s-t+i}(\sec \mathcal{K}(\Gamma^s,i)) \qquad \text{by 4.4} \\
&\approx \pi_0(\Omega^{s-t+i} \sec \mathcal{K}(\Gamma^s,i)) \\
&\approx \pi_0 \sec[\Omega_B^{s-t+i} \mathcal{K}(\Gamma^s,i)] \\
&\approx \pi_0 \sec \mathcal{K}(\Gamma^s,t-s) \\
&\approx \mathcal{H}^{t-s}(B;\Gamma^s)
\end{aligned}
$$

since $\mathcal{K}(\Gamma^s, t-s)$ represents $\mathcal{H}^{t-s}(\ \ ;\Gamma^s)$ over B. These isomorphisms hold for all sufficiently large i and, using the relation between the k-invariants of $\mathcal{M}_i$ and $\sigma\mathcal{M}_i$ (3.3(i)), one verifies that they are compatible with the morphisms in the direct system defining $Q_{s-t}(s,s+1)$. Hence

$$E_1^{s,t} = Q_{s-t}(s,s+1) \approx \mathcal{H}^{t-s}(B;\Gamma^s).$$

We have also shown that ${}_iQ_n(s,s+1) \to Q_n(s,s+1)$ is an isomorphism for $n+i \geq 0$, $s+1 \leq h_i$. (ii) now follows from this by an inductive five-lemma argument.

Since ${}_iQ_n(0,\infty) \approx \pi_{i+n}\sec\sigma^i p$, (iii) is exactly the remark after 1.7.

4.11 <u>Proposition</u>. Suppose B is finite-dimensional. Then the spectral sequence converges strongly: the filtration on the abutment is finite in each degree.

Proof. The filtration is given by

$F^mQ_n(0,\infty) = \ker[Q_n(0,\infty) \to Q_n(0,m)]$. Choose $i > \max(-n, n+\dim B - 2c)$.

Then
$$\begin{aligned} F^{h_i}Q_n(0,\infty) &\approx \ker\,[{}_iQ_n(0,\infty) \to {}_iQ_n(0,h_i)] \quad \text{by 4.10} \\ &\approx \ker\,[\pi_{i+n}(\sec\sigma^i X) \to \pi_{i+n}(\sec Y_0^{h_i})\,]. \end{aligned}$$

Consider the map $\sigma^i X \to Y_0^{h_i}$ of fibrations over B (4.3). This induces a $(2c + 2i + 1)$-equivalence on fibres, since $\mathcal{M}_i$ kills homotopy groups up to $\pi_{2c+2i}(\Sigma^i F)$. Hence $\sec \sigma^i X \to \sec Y_0^{h_i}$ is a $(2c + 2i + 1 - \dim B)$-equivalence by 1.1 (using mapping cylinders). Thus by our assumption on i, $\pi_{i+n}(\sec \sigma^i X) \approx \pi_{i+n}(\sec Y_0^{h_i})$ and so $F^{h_i} Q_n(0,\infty) = 0$. But trivially $F^0 Q_n(0,\infty) = Q_n(0,\infty)$ because $\sec Y_0^0$ is a point.

4.12 Twisted cohomology operations with local coefficients. These operations are the local coefficient analogues of those in [12], [24]. We deal here with primary operations, although it is easy enough to define n'th order operations by a universal example method.

Let Γ, Δ be graded $\pi_1 B$ - modules. We call an element $\alpha \in \mathcal{H}^n(\mathcal{K}(\Gamma,m),B;\Delta)$ a universal example for an operation of type $(\Gamma, m; \Delta, n)$. α can be represented by a map $\mathcal{K}(\Gamma, m) \to \mathcal{K}(\Delta, n)$ which preserves both the projection onto B and the zero section. We can compose this map with the map $(X, A) \to (\mathcal{K}(\Gamma,m), B)$ representing any $\xi \in \mathcal{H}^m(X,A;\Gamma)$. This process defines a natural transformation $\mathcal{H}^m(\ ,\ ;\Gamma) \to \mathcal{H}^n(\ ,\ ;\Delta)$ called the operation defined by α.

If we apply the functor Ω_B of 4.9 to the map $\mathcal{K}(\Gamma,m) \to \mathcal{K}(\Delta,n)$ representing α, we obtain a map representing an operation $\Omega\alpha$ of type $(\Gamma, m-1; \Delta, n-1)$. We call $\Omega\alpha$ the suspension of α. It is an additive operation, and has the expected relationship with the suspension homomorphism $S : \mathcal{H}^{m-1}(X;\Gamma) \to \mathcal{H}^{m}(\sigma X, B; \Gamma)$ of 3.1; namely $\alpha(S\xi) = S[(\Omega\alpha)(\xi)]$ for $\xi \in \mathcal{H}^{m-1}(X;\Gamma)$.

We define a stable operation to be a sequence $\{\alpha_i\}_{i\in Z}$, where α_i is of type $(\Gamma, m+1; \Delta, n+i)$ and $\alpha_i = \Omega\alpha_{i+1}$.

We can now identify the $d_1^{s,t}$ in the spectral sequence.

4.13 Proposition. For fixed s and variable t, the differentials $d_1^{s,t}: \mathcal{H}^{t-s}(B;\Gamma^s) \to \mathcal{H}^{t-s+1}(B;\Gamma^{s+1})$ form a stable twisted cohomology operation in the above sense.

If $s + 2 \leq h_1$, the operation $d_1^{s,s+i} : \mathcal{H}^{i}(B;\Gamma^s) \to \mathcal{H}^{i+1}(B;\Gamma^{s+1})$ is represented by the row

$$\mathcal{K}(\Gamma^s,i) \simeq Y_s^{s+1} \subset \dots \subset Y_0^{s+1} \xrightarrow{k^{s+1}} \mathcal{K}(\Gamma^{s+1},i+1)$$

in diagram 4.3.

Proof. By definition $d_1^{s,t}$ is the boundary map $\partial : Q_{s-t}(s,s+1) \to Q_{s-t-1}(s+1,s+2)$ of Q_*. If i satisfies $s - t + i \geq 0$ and $s + 2 \leq h_i$, we can calculate this from the system ${}_iQ_*$: it is given by the homotopy boundary $\partial : \pi_{s-t+i}(\sec Y_s^{s+1}) \to \pi_{s-t+i-1}(\sec Y_{s+1}^{s+2})$ in the fibration $\sec Y_{s+1}^{s+2} \to \sec Y_s^{s+2} \to \sec Y_s^{s+1}$ of 4.5(i). By 4.5(ii) this fibration sequence can be continued to the right by the map $\sec Y_s^{s+1} \to \mathcal{K}(\Gamma^{s+1}, i+1)$ induced by the map in 4.13. Thereby the homotopy boundary required is identified with the induced homomorphism $\pi_{s-t+i}(\sec Y_s^{s+1}) \to \pi_{s-t+i}[\sec \mathcal{K}(\Gamma^{s+1}, i+1)]$. Using the isomorphism of functors $\pi_{s-t+i} \sec (\ \ ) \approx \pi_0 \sec \Omega_B^{s-t+i}(\ \ )$ to identify the groups with cohomology groups as before, we find that $d_1^{s,t}$ is the $(s - t + i)$-fold suspension of the operation represented by $\mathcal{K}(\Gamma^s,i) \simeq Y_s^{s+1} \to \mathcal{K}(\Gamma^{s+1},i+1)$. This proves 4.13.

It is possible to identify $d_r^{s,t}$ with a suitably-defined r'th order stable twisted operation, but this seems to be little help in practice.

4.14 <u>Naturality</u>. We show that the spectral sequence is natural (contravariant) with respect to induced fibrations. Let $f : C \to B$ be a map; denote the fibration induced by f from p by $f^*p : X \times_B C \to C$.

From any Eilenberg-MacLane object $\mathcal{K}(\Gamma,n) \to B$ over B, we obtain an Eilenberg-MacLane object $\mathcal{K}(\Gamma,n) \times_B C \to C$ over C (corresponding to the induced $\pi_1 C$-module structure on Γ) by forming the fibre product. It follows that any MPT for p yields an MPT for f^*p : we only have to apply the functor $- \times_B C$ to each space in the tower. The map $f : C \to B$ is covered by a canonical map of MPT's (projection of each fibre product). The functor $- \times_B C$ commutes with σ, so an SMPT for p induces an SMPT for f^*p. The map of SMPT's covering $f : C \to B$ induces a map from the $H(p,q)$-system of p to that of f^*p (sections yield induced sections) and hence a morphism of spectral sequences.

4.15 <u>A practical method of determining $d_1^{s,t}$</u>. We show how $d_1^{s,t}$ can be calculated from the tower $\mathcal{M}_0$ when $s + 1 < h_0$ (h_0 is the height of $\mathcal{M}_0$). This is the method we use in calculations. The approach was suggested by a method in [23].

Let $p^{(2)}$ be the induced fibration

$$\begin{array}{ccc} X \times_B X & \to & X \\ p^{(2)}\downarrow & & p\downarrow \\ X & \xrightarrow{p} & B \end{array}$$

and let $X \to Y_0^{h_0} \to \cdots \to Y_0^0 = B$ be the tower $\mathcal{M}_0$ for p. Then we have the induced tower

$$X \times_B X \to Y_0^{h_0} \times_B X \to \cdots \to Y_0^0 \times_B X = X$$

for $p^{(2)}$. Since $p^{(2)}$ has a standard section $x \to (x,x)$, we can construct a triangular diagram from this tower (whereas we can not from $\mathcal{M}_0$). Part of this diagram is

$$\begin{array}{lccccc} & (\Gamma^s,0) \times_B X & \subset & Y_0^{s+1} \times_B X & \xrightarrow{k^{s+1} \times 1} & \mathcal{K}(\Gamma^{s+1},1) \times_B X \\ (4.16) & \downarrow & & \downarrow & & \\ & X & \subset & Y_0^s \times_B X & & \end{array}$$

where k^{s+1} is a k-invariant of $\mathcal{M}_0$. The composite $\mathcal{K}(\Gamma^s,0) \times_B X \to \mathcal{K}(\Gamma^{s+1},1) \times_B X$ is a map of Eilenberg-MacLane objects over X which, by 4.13, represents a cohomology operation whose suspensions give the $d_1^{s,t}$ in the (induced) spectral sequence for $p^{(2)}$. That is, the universal example giving the $d_1^{s,t}$ in the induced spectral sequence is the element of $\mathcal{H}^1(\mathcal{K}(\Gamma^s,0) \times_B X, X; \Gamma^{s+1})$ induced from $k^{s+1} \in \mathcal{H}^1(Y_0^{s+1}; \Gamma^{s+1})$ by the map

$$\mathcal{K}(\Gamma^s,0) \times_B X \to Y_0^{s+1} \times_B X \xrightarrow{\text{proj}} Y_0^{s+1} .$$

Projecting $Y_0^{s+1} \times_B X$, $Y_0^s \times_B X$ on to their first factors, we obtain from (4.16) a diagram

$$(4.17) \qquad \begin{array}{ccc} \mathcal{K}(\Gamma^s,0) \times_B X & \xrightarrow{\nu} & Y_0^{s+1} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Y_0^s \end{array}$$

which is an induced fibration square since (4.16) is. This diagram is an adaptation of the diagram used by Thomas in ([23], pp.17-18). Using mapping cylinders, we may regard ν, ϕ as inclusions and write down the relative Serre exact sequence for the fibration-pair in (4.17)

$$\cdots \to \mathcal{H}^i(Y_0^{s+1};\Gamma^{s+1}) \xrightarrow{\nu^*} \mathcal{H}^i(\mathcal{K}(\Gamma^s,0) \times_B X;\Gamma^{s+1}) \xrightarrow{\tau_0} \mathcal{H}^{i+1}(Y_0^s,X;\Gamma^{s+1}) \to \cdots$$

where τ_0 is the relative transgression as in [23]. (The sequence exists only for i in a certain range depending on Γ^s, Γ^{s+1}). We have proved :

4.18 Proposition. The image of the k-invariant $k^{s+1} \in \mathcal{H}^1(Y_0^{s+1}; \Gamma^{s+1})$ under the homomorphism ν^* lies in $\mathcal{H}^1(\mathcal{K}(\Gamma^s,0) \times_B X, X; \Gamma^{s+1})$ and is a universal example for the stable operation $d_1^{s,*}$ in the spectral sequence for $p^{(2)}$.

We still have to obtain the differentials in the original spectral sequence from those in the induced one. The original $d_1^{s,*}$ have a universal example in $\mathcal{H}^1(\mathcal{K}(\Gamma^s,0),B; \Gamma^{s+1})$ which, by naturality of the SMPT , maps to the universal example for the induced $d_1^{s,*}$ under the homomorphism π^* induced by the projection $\pi : (\mathcal{K}(\Gamma^s,0) \times_B X, X) \to (\mathcal{K}(\Gamma^s,0),B)$. In the stable range of dimensions $\leq 2c+2$, π^* is a monomorphism because both the fibre F of π and the base-pair $(\mathcal{K}(\Gamma^s,0),B)$ are c-connected. Combining this result with 4.18, we have a complete method for obtaining $d_1^{s,t}$ from k^{s+1} in the stable range (where all our calculations will take place).

Yet more is true. In all our 2-primary calculations, ν^* will be a monomorphism. Hence by reversing the above procedure one can deduce k^{s+1} from $d_1^{s,t}$. For this reason tables of k-invariants have been omitted from §5.

We sum up the results of this section.

4.19. Theorem. Let $p : X \to B$ be a fibration with c-connected fibre F $(c \geq 1)$, where B is a connected CW complex. Let $\{\mathcal{M}_i\}_{i \geq 0}$ be an SMPT for p with graded coefficient $\pi_1 B$-modules $\{\Gamma^0, \Gamma^1, \ldots\}$. Then

(i) there is a spectral sequence $E_1^{s,t} \approx \mathcal{H}^{t-s}(B;\Gamma^s) \Rightarrow \pi_*^\sigma(p)$ $(s \geq 0,\ t \in Z)$ with total degree $s - t$, and $d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t+r+1}$

(ii) $d_1^{s,t} : \mathcal{H}^{t-s}(B;\Gamma^s) \to \mathcal{H}^{t-s+1}(B;\Gamma^{s+1})$ is a stable twisted cohomology operation : $d_1^{s,t}$ is the suspension of $d_1^{s,t+1}$

(iii) the filtration on $\pi_*^\sigma(p)$ is decreasing:

$$\pi_n^\sigma = F^0\pi_n^\sigma \supset \ldots \supset F^s\pi_n^\sigma \supset F^{s+1}\pi_n^\sigma \supset \ldots \text{ where } E_\infty^{s,s-n} \approx F^s\pi_n^\sigma / F^{s+1}\pi_n^\sigma .$$

(iv) if B is finite-dimensional the filtration is finite in each degree and the spectral sequence is strongly convergent

(v) the spectral sequence is natural with respect to induced fibrations.

4.20 If we choose $\{\mathcal{M}_i\}$ to be the canonical stable Postnikov tower of 3.7, then $E_1^{s,t} \approx H^t(B;\pi_s^\Sigma F)$ where

$\pi_s^{\Sigma} F$ is the s'th stable homotopy group of the fibre and the local coefficients are given by the usual action of $\pi_1 B$ ($\approx \pi_1 E$) on this group. This special case is more perspicuous than that of 4.19, but it is less useful for calculations.

§5 Calculations for the fibrations $BO_n \to BO$

5.0 We now compute explicit MPT's for the universal Stiefel bundles $O/O_n \to BO_n \to BO$. The techniques we use are those of [23], suitably modified to take account of the local coefficients involved in our case. By 3.6, these MPT's extend to SMPT's and hence yield spectral sequences by 4.19. Our calculations yield enough of the E_1-term to facilitate calculation of $\pi_0^\sigma(p)$ for any O/O_n-bundle p with base dimension $\leq n+5$, provided the necessary differentials can be computed. The relevant $E_1^{s,t}$ and $d_1^{s,t}$ are tabulated in 5.8. In 5.12 we make our principal application: we obtain information about the number of immersions of real projective n-space P^n in R^{2n-k} $(k \leq 5)$ when these exist.

5.1 Theorem Let $n \geq 7$. Then the universal Stiefel bundle $O/O_n \to BO_n \to BO$ has an MPT (with sensible coefficients) of height ≤ 4 which kills homotopy groups up to $\pi_{n+5}(O/O_n)$ and leaves $\pi_i(O/O_n)$ unaltered for $i > n+5$.

The proof of 5.1 is carried out separately for each of the eight possible residue classes of n mod 8. We give the details in the case $n \equiv 6$, and tabulate the results of the computation in the other cases.

Let $n = 8s + 6$, $s \geq 1$. Then the fibre O/O_n is $(n-1)$-connected, and according to [18] its six following homotopy groups are $\pi_n(O/O_n) \approx Z$, $\pi_{n+1} \approx Z_4$, $\pi_{n+2} \approx 0$, $\pi_{n+3} \approx Z_{12}$, $\pi_{n+4} \approx 0$, $\pi_{n+5} \approx Z_8$.

The first stage in the MPT for $p : BO_n \to BO$ is constructed to kill $\pi_n(O/O_n)$ and part of $\pi_{n+1}(O/O_n)$. Let $\tilde{Z}$ denote the group Z with its non-trivial $\pi_1(BO)$-module structure ($\pi_1(BO) \approx Z_2$). Then we have Stiefel-Whitney classes $W_{n+1} \in H^{n+1}(BO; \tilde{Z})$, $w_{n+2} \in H^{n+2}(BO; Z_2)$. W_{n+1} is the transgression of the fundamental class $-\gamma_1 \in H^n(O/O_n; Z)$ in the fibration p. Since $p^* w_{n+2} \in H^{n+2}(BO_n; Z_2)$ is zero, w_{n+2} is the transgression of some class $-\gamma_2 \in H^{n+1}(O/O_n; Z_2)$. The pair (W_{n+1}, w_{n+2}) is represented by a map from BO into a certain generalised Eilenberg-MacLane object over BO. In the notation of 2.1, this object is $K_{BO}(\tilde{Z}, n+1) \times K(Z_2, n+2)$. By 2.7 there is a map of fibrations

$$\begin{tikzcd}
O/O_n \arrow[r] \arrow[d] & K(Z, n) \times K(Z_2, n+1) \arrow[d] \\
BO_n \arrow[r] \arrow[d] & P_{BO}[K_{BO}(\tilde{Z}, n+1) \times K(Z_2, n+2)] \arrow[d] \\
BO \arrow[r, "(W_{n+1}, w_{n+2})"] & K_{BO}(\tilde{Z}, n+1) \times K(Z_2, n+2)
\end{tikzcd}$$

such that $O/O_n \to K(Z,n) \times K(Z_2,n+1)$ represents the pair of classes (γ_1,γ_2). Let $p_1:X^1 \to BO$ be the fibration induced by the map (W_{n+1},w_{n+2}) from the universal fibration over $K_{BO}(\tilde{Z},n+1) \times K(Z_2,n+2)$. Then $p:BO_n \to BO$ factorises as $BO_n \xrightarrow{q_1} X^1 \xrightarrow{p_1} BO$. To show this is an elementary factorisation in the sense of 2.8, we must check that $O/O_n \to K(Z,n) \times K(Z_2,n+1)$ induces epimorphisms of homotopy groups; i.e. that (γ_1,γ_2) is a spherical set. However, this is an immediate consequence of 6.1.3 of [13]. Therefore we can take $X^1 \to BO$ as the first stage of our MPT, and (W_{n+1},w_{n+2}) as the first k-invariant k^0.

We have a diagram

$$\begin{array}{lcccl}
 & V_1 \to & O/O_n & \to & BO_n \\
 & & \downarrow{\scriptstyle(\gamma_1,\gamma_2)} & & \downarrow{\scriptstyle q_1} \\
(5.2) & & K(Z,n) \times K(Z_2,n+1) & \to & X^1 \\
 & & & & \downarrow{\scriptstyle p_1} \\
 & & & & BO \xrightarrow{k^0} K_{BO}(\tilde{Z},n+1) \times K(Z_2,n+2)
\end{array}$$

where V_1 is the fibre of the map q_1. From the exact sequences of 2.8, V_1 is n-connected and $\pi_{n+1}V_1 \approx Z_2$, $\pi_{n+2}V_1 \approx 0$, $\pi_{n+3}V_1 \approx Z_{12}$, $\pi_{n+4}V_1 \approx 0$, $\pi_{n+5}V_1 \approx Z_8$.

The next k-invariant will lie in $H^*(X^1)$. We obtain it by a modification of a method of [23]. We first write down the diagram (4.17) for the first stage of our MPT :

$$(5.3) \qquad \begin{array}{ccc} K_{BO_n}(\widetilde{Z},n) \times K(Z_2,n+1) & \xrightarrow{\nu} & X^1 \\ \downarrow & & \downarrow p_1 \\ BO_n & \rightarrow & BO \end{array}$$

where $K_{BO_n}(\widetilde{Z},n) = K_{BO}(\widetilde{Z},n) \times_{BO} BO_n$ is the Eilenberg-MacLane object over BO_n corresponding to the non-trivial action of $\pi_1 BO_n$ on Z, and the vertical map on the left is the natural projection of the first factor on to BO_n.

We now need information about the mod 2 and mod 3 cohomology of $K_{BO_n}(\widetilde{Z},n)$. This is given in the following two lemmas.

5.4 Lemma. Let $m \geq 1$, $n \geq 1$. Then

(i) there is a canonical isomorphism of Z_2-algebras

$$H^*(K(Z,m);Z_2) \otimes H^*(BO_n;Z_2) \approx H^*(K_{BO_n}(\widetilde{Z},m);Z_2) .$$

(ii) Let ι be the fundamental class mod 2 of $K(Z,m)$. Then with the above identification of $H^*(K_{BO_n}(\tilde{Z},m);Z_2)$, the action of the Steenrod algebra A_2 on this algebra is determined by

(a) A_2 acts in the usual way on the subalgebra $H^*(BO_n;Z_2)$

(b) $\alpha(\iota \otimes 1) = (\alpha\iota) \otimes 1$ if $\alpha \in A_2$ is any Cartan basis element not involving Sq^1

(c) $Sq^1(\iota \otimes 1) = \iota \otimes w_1$.

Proof. (i) By definition of $K_{BO_n}(\tilde{Z},m)$ (2.1) there is a sectioned fibration $K(Z,m) \to K_{BO_n}(\tilde{Z},m) \overset{\pi}{\to} BO_n$. Let $\hat{\iota}_2$ be the mod 2 reduction of the (local coefficient) universal class of $K_{BO_n}(\tilde{Z},m)$. We first define a Z_2-algebra homomorphism $\phi: H^*(K(Z,m);Z_2) \to H^*(K_{BO_n}(\tilde{Z},m);Z_2)$. If $\alpha \in A_2$ is any Cartan basis element of excess $< m$ not involving Sq^1, set $\phi(\alpha\iota) = \alpha\hat{\iota}_2$. Such $\alpha\iota$ generate $H^*(K(Z,m);Z_2)$ freely as a polynomial algebra, so ϕ extends to a unique homomorphism of algebras. If $i: K(Z,m) \subset K_{BO_n}(\tilde{Z},m)$ is the inclusion of the fibre, $i^*\phi = 1$. Now define $H^*(K(Z,m);Z_2) \otimes H^*(BO_n;Z_2) \to H^*(K_{BO_n}(\tilde{Z},m);Z_2)$ by $x \otimes y \to \phi(x).\pi^*y$. By the Leray-Hirsch theorem this is an isomorphism, and it clearly preserves products.

(ii) (a) and (b) are evident from the definition of the isomorphism. To prove (c), we note that $Sq^1(\iota \otimes 1)$ is non-zero: for the spectral sequence of the fibration $K(Z,m) \to K_{BO_n}(\tilde{Z},m) \to BO_n$ with (simple) Z_4 coefficients shows that $\iota \otimes 1 = \hat{\iota}_2$ is not the reduction of a Z_4-class. Yet $Sq^1(\iota \otimes 1)$ restricts to zero on the zero-section $BO_n \subset K_{BO_n}(\tilde{Z},m)$. The only non-zero class with this property is $\iota \otimes w_1$.

. 5.5 <u>Lemma</u>. Let $m,n \geq 2$. Let $\tilde{Z}_3$ denote the group Z_3 with the non-trivial π_1BO_n-module structure, and let $\hat{\iota}_3 \in H^m(K_{BO_n}(\tilde{Z},m);\tilde{Z}_3)$ be the mod 3 reduction of the universal class. Then

$$H^{m+4}(K_{BO_n}(\tilde{Z},m),BO_n;\tilde{Z}_3) \approx Z_3 + Z_3$$

with generators $\mathcal{P}^1_3\,\hat{\iota}_3$, $p_1.\,\hat{\iota}_3$, where $\mathcal{P}^1_3$ is the reduced power operation extended to local coefficients as in [6], and $p_1 \in H^4(BO_n;Z_3)$ is the first Pontryagin class.

Proof. This is immediate from the spectral sequence (2.4) of the fibration $K(Z,m) \to K_{BO_n}(\tilde{Z},m) \to BO_n$ with non-trivial Z_3 coefficients.

We now return to the MPT for $BO_n \to BO$, $n = 8s + 6$. To obtain a mod 2 k-invariant at the second stage, we consider the relative Serre exact sequence of the fibration-pair in diagram 5.3 :

$$\cdots \to H^i(X^1;Z_2) \overset{\nu^*}{\to} H^i(K_{BO_n}(\tilde{Z},n) \times K(Z_2,n+1);Z_2) \overset{\tau}{\to} H^{i+1}(BO,BO_n;Z_2) \to \cdots$$

which is valid as far as $H^{2n}(K_{BO_n}(\tilde{Z},n) \times K(Z_2,n+1);Z_2)$. Now $BO_n \to BO$ induces an epimorphism in mod 2 cohomology in all dimensions. For $i \leq n + 7$, all elements of the kernel of $p^*:H^i(BO;Z_2) \to H^i(BO_n;Z_2)$ are already in the kernel of $p_1^* : H^i(BO;Z_2) \to H^i(X^k;Z_2)$ (because $w_{n+1}, w_{n+2} \in \ker p_1^*$, whence by Wu's formulae $w_{n+3}, \ldots, w_{n+7}$ are also). It follows as on p.16 of [23] that the above exact sequence gives exact sequences

$$0 \to H^i(X^1;Z_2) \overset{\nu^*}{\to} H^i(K_{BO_n}(\tilde{Z},n) \times K(Z_2,n+1);Z_2) \overset{\tau_1}{\to} H^{i+1}(BO;Z_2)$$

for $i \leq n+7$, where τ_1 is the composite of τ with $H^{i+1}(BO, BO_n; Z_2) \to H^{i+1}(BO; Z_2)$.

The k-invariant we seek must lie in the kernel of $q_1^* : H^*(X^1; Z_2) \to H^*(BO_n; Z_2)$ (diagram 5.2). This kernel can be computed from the above exact sequence by the methods of ([23],§6). For this one needs to know the structure of $H^*(K_{BO_n}(\widetilde{Z}, n) \times K(Z_2, n+1); Z_2)$ as an algebra over A_2 : but this is known from 5.4. We can identify $H^*(K_{BO_n}(\widetilde{Z}, n) \times K(Z_2, n+1); Z_2)$ with $H^*BO_n \otimes H^*K(Z,n) \otimes H^*K(Z_2, n+1)$ as a Z_2-algebra: let ι_1, ι_2 denote the elements of this corresponding to the fundamental classes of $K(Z,n)$, $K(Z_2,n+1)$. Then a tedious computation shows that $\ker q_1^* \subset H^*(X^1; Z_2)$ is generated over A_2 in dimensions $\leq n+6$ by three elements k_1^1, k_2^1, k_3^1 and their products with elements of $H^*(BO_n; Z_2)$, where

$$\nu^* k_1^1 = Sq^2 \iota_1 + w_2 . \iota_1 + Sq^1 \iota_2$$

$$\nu^* k_2^1 = Sq^4 \iota_1 + w_4 . \iota_1 + Sq^3 \iota_2 + w_1 . Sq^2 \iota_2 + w_2 . Sq^1 \iota_2$$

$$\nu^* k_3^1 = Sq^4 Sq^1 \iota_2 + (w_1^2 + w_2) . Sq^3 \iota_2 + (w_1 w_2 + w_3) Sq^2 \iota_2 + (w_2^2 + w_4) Sq^1 \iota_2 .$$

Consider the fibration $V_1 \to BO_n \xrightarrow{q_1} X^1$ of diagram 5.2. Since $k_1^1, k_2^1, k_3^1 \in \ker q_1^*$, there are classes $-\delta_1 \in H^{n+1}(V_1;Z_2)$, $-\delta_2 \in H^{n+3}(V_1;Z_2)$, $-\delta_3 \in H^{n+5}(V_1;Z_2)$, $-\delta_3 \in H^{n+5}(V_1;Z_2)$ which transgress to k_1^1, k_2^1, k_3^1. The theorems of [23] show that δ_1 and δ_2 are spherical classes (since Sq^2, Sq^4 are irreducible operations of type 1 in the terminology of [23]) and a short computation with the Postnikov system of V_1 shows δ_3 is also spherical. Therefore k_1^1, k_2^1, k_3^1 can be used as components of the second k-invariant.

We now seek a mod 3 component for the second k-invariant to kill the 3-primary component of $\pi_{n+3}V_1 \approx Z_{12}$. The Serre exact sequence of the fibration-pair 5.3 with non-trivial Z_3 coefficients is

$$\ldots \to H^i(X^1;\tilde{Z}_3) \xrightarrow{\nu^*} H^i(K_{BO_n}(\tilde{Z},n) \times K(Z_2,n+1);\tilde{Z}_3) \xrightarrow{\tau} H^{i+1}(BO,BO_n;\tilde{Z}_3) \to \ldots$$

$(i < 2n)$. Now $H^{n+1}(BO,BO_n;\tilde{Z}_3) \approx Z_3$, generated by the coboundary $\delta\chi_3$ of the mod 3 twisted Euler class $\chi_3 \in H^n(BO_n;\tilde{Z}_3)$; and the mod 3 reduction $\hat{\iota}_3 \in H^n(K_{BO_n}(\tilde{Z},n);\tilde{Z}_3)$ of the universal class satisfies $\tau(\hat{\iota}_3) = \pm\,\delta\chi_3$. We consider the above sequence with

$i = n + 4$. After Lemma 5.5, we have free generators $\mathcal{P}_3^1\,\hat{\iota}_3,\ p_1\cdot\hat{\iota}_3$ for $H^{n+4}(K_{BO_n}(\tilde{Z},n) \times K(Z_2,n+1),BO_n;\tilde{Z}_3) \subset H^{n+4}(K_{BO_n}(\tilde{Z},n) \times K(Z_2,n+1);\tilde{Z}_3)$

(since $K(Z_2, n+1)$ has no mod 3 cohomology). Since $\mathcal{P}_3^1\,\chi_3 = p_1\cdot\chi_3$, we have $\tau(\mathcal{P}_3^1\,\hat{\iota}_3 - p_1\cdot\hat{\iota}_3) = (\mathcal{P}_3^1 - p_1)(\pm\delta\chi_3) = 0$. Hence by exactness there is a class $k_4^1 \in H^{n+4}(X_1;\tilde{Z}_3)$ with $\nu^* k_4^1 = \mathcal{P}_3^1\,\hat{\iota}_3 - p_1\cdot\hat{\iota}_3$. The theorems of [23] guarantee that k_4^1 is the transgression of a spherical class $-\delta_4 \in H^{n+3}(V_1;Z_3)$ in the fibration $V_1 \to BO_n \to X^1$.

Applying 2.7 to the generalised class $(\delta_1,\delta_2,\delta_3,\delta_4) \in \mathcal{H}^*(V_1)$, we obtain an elementary factorisation $BO_n \to X^2 \to X^1$ of the fibration $BO_n \to X^1$ with k-invariant $k^1 = (k_1^1,\ k_2^1,\ k_3^1,\ k_4^1)$. This gives a diagram

$$\begin{tikzcd}
V_2 \arrow[r] & V_1 \arrow[r] \arrow[d, "{(\delta_1,\delta_2,\delta_3,\delta_4)}"'] & BO_n \arrow[d, "q_2"] & \\
& K_{n+1} \times K_{n+3} \times K_{n+5} \times K(Z_3,n+3) \arrow[r] & X^2 \arrow[d, "p_2"] & \\
& & X^1 \arrow[r, "k^1"] & K_{n+2} \times K_{n+4} \times K_{n+6} \times K_{BO}(\tilde{Z}_3,n+4)
\end{tikzcd}$$

where (as hereafter) K_i denotes $K(Z_2,i)$, and V_2 is the fibre of $BO_n \to X^2$. From the exact sequences of 2.8, V_2 is (n+2)-connected, $\pi_{n+3}V_2 \approx Z_2$, $\pi_{n+4}V_2 \sim 0$, $\pi_{n+5}V_2 \sim Z_4$. This completes the second stage.

To obtain the k-invariant for the third stage, we consider the diagram (4.17) for the stage $X^2 \to X^1$:

$$\begin{array}{ccc} K_{n+1} \times K_{n+3} \times K_{n+5} \times K_{BO_n}(Z_3,n+3) & \xrightarrow{\nu} & X^2 \\ \downarrow & & \downarrow \\ BO_n & \longrightarrow & X^1 \end{array} .$$

As before one proves that the resulting sequence

$$0 \to H^i(X^2;Z_2) \xrightarrow{\nu^*} H^i(K_{n+1} \times K_{n+3} \times K_{n+5} \times K_{BO_n}(Z_3,n+3);Z_2) \xrightarrow{\tau_1} H^{i+1}(X^1;Z_2)$$

is exact for $i \leq n + 6$. Our k-invariant will lie in the kernel of $q_2^* : H^*(X^2;Z_2) \to H^*(BO_n;Z_2)$. Let $\kappa_1, \kappa_2, \kappa_3$ be the fundamental classes of $K_{n+1}, K_{n+3}, K_{n+5}$. Then calculation shows that $\ker q_2^* \subset H^*(X^2;Z_2)$ is generated over A_2 in dimensions $\leq n + 6$ by two elements k_1^2, k_2^2 and their products with elements of $H^*(BO_n;Z_2)$, where

$$\nu^* k_1^2 = Sq^1 \kappa_2 + w_1 . \kappa_2 + Sq^2 Sq^1 \kappa_1 + w_1^2 . Sq^1 \kappa_1 + w_3 . \kappa_1$$

$$v^* k_2^2 = Sq^1 \kappa_3 + Sq^2 Sq^3 \kappa_1 + w_2 . Sq^2 Sq^1 \kappa_1 + (w_1 w_3 + w_2^2 + w_1^2 w_2) \quad Sq^1 \kappa_1$$
$$+ (w_1 w_4 + w_5) . \kappa_1 .$$

Since Sq^1 is an irreducible operation, the theorems of [23] show that there are spherical classes $-\epsilon_1 \in H^{n+3}(V_2;Z_2)$, $-\epsilon_2 \in H^{n+5}(V_2;Z_2)$ which transgress to k_1^2, k_2^2 in the fibration $V_2 \to BO_n \to X^2$.
So we take $k^2 = (k_1^2, k_2^2)$ as the third k-invariant, and construct as before a diagram

$$\begin{tikzcd}
V_3 \arrow[r] & V_2 \arrow[r] \arrow[d, "(\epsilon_1, \epsilon_2)"'] & BO_n \arrow[d, "q_3"] & \\
& K_{n+3} \times K_{n+5} \arrow[r] & X^3 \arrow[d] & \\
& & X^2 \arrow[r, "k^2"] & K_{n+4} \times K_{n+6} .
\end{tikzcd}$$

Here the fibre V_3 of q_3 is $(n+4)$-connected, and $\pi_{n+5} V_3 \approx Z_2$.

For the fourth k-invariant we choose the characteristic class $k^3 \in H^{n+6}(X^3;\, Z_2)$ of the fibration $V_3 \to BO_n \to X^3$. This kills $\pi_{n+5}V_3$ and completes the tower. Computation shows that under the homomorphism

$$\nu^*: H^{n+6}(X^3;Z_2) \to H^{n+6}(BO_n \times K_{n+3} \times K_{n+5}\ ;\ Z_2)$$

of 4.17, k^3 maps to

$$Sq^1\mu_2 + Sq^2Sq^1\mu_1 + w_1Sq^2\mu_1 + w_2Sq^1\mu_1 + (w_1w_2 + w_1^3)\,\mu_1 \tag{5.6}$$

where μ_1, μ_2 are the fundamental classes of K_{n+3}, K_{n+5} .

The constructive proof of 5.1 in the case $n = 8s + 6$ is complete. The tower is

$$\begin{array}{lll}
BO_n & & \\
\downarrow & & \\
X^4 & & \\
\downarrow & & \\
X^3 & \xrightarrow{k^3} & K_{n+6} \\
\downarrow & & \\
X^2 & \xrightarrow{k^2} & K_{n+4} \times K_{n+6} \\
\downarrow & & \\
X^1 & \xrightarrow{k^1} & K_{n+2} \times K_{n+4} \times K_{n+6} \times K_{BO}(\mathbb{Z}_3, n+4) \\
\downarrow & & \\
BO & \xrightarrow{k^0} & K_{BO}(\mathbb{Z}, n+1) \times K_{n+2} \quad .
\end{array}$$

5.7 By Proposition 3.6, the MPT for $p: BO_n \to BO$ constructed in 5.1 can be extended to an SMPT. We choose one such extension for each n; for the purpose of our computations it will not matter what choice is made.

Let ξ be a stable real vector bundle over a complex B, and let $f: B \to BO$ be a classifying map for ξ. Then the bundle associated to ξ with fibre O/O_n is the map $p': Y \to B$ in the pullback diagram

$$\begin{array}{ccc} Y & \to & BO_n \\ {\scriptstyle p'}\downarrow & & \downarrow {\scriptstyle p} \\ B & \xrightarrow{f} & BO \end{array} .$$

The SMPT for p induces an SMPT for p', by 4.14. We consider the resulting spectral sequence for $\pi_*^\sigma(p')$. According to 4.19, the E_1-term can be read off from the coefficient modules of the SMPT. The first few coefficient modules are those of the MPT of 5.1. Hence when $n = 8s + 6$, for example,

$$E_1^{0,q} \approx H^{q+n}(B;\widetilde{Z}) + H^{q+n+1}(B;Z_2)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$E_1^{3,q} \approx H^{q+n+2}(B;Z_2)$$

where $\tilde{Z}$ is the local system of integer coefficients on B induced by f from the non-trivial system on BO.

By 4.18 and the remark following it, the corresponding $d_1^{s,t}$ in the spectral sequence can be obtained immediately from the images of the k-invariants k^i under the homomorphism ν^* of the appropriate Serre exact sequence. By naturality, these can be computed from the MPT for the universal example $BO_n \to BO$. Fortunately, these $\nu^* k^i$ are obtained as a matter of course in the construction of the MPT, as the proof of 5.1 illustrates. For instance, in the case $n = 8s + 6$, (5.6) shows that
$d_1^{2,q}: H^{q+n+1}(B;Z_2) + H^{q+n+3}(B;Z_2) \to H^{q+n+4}(B;Z_2)$ is given by the matrix

$$[\, Sq^2Sq^1 + w_1Sq^2 + w_2Sq^1 + w_1w_2 + w_1^3 , \quad Sq^1 \,]$$

where w_i denotes $w_i(\xi)$.

5.8 <u>Tables of spectral sequences</u>. As before, ξ is a stable real vector bundle over a complex B. We give below that part of the E_1-term of the spectral sequence 4.19 for the associated bundle with fibre $0/0_n$ $(n \geq 7)$ which can be inferred from the MPT's of 5.1.

The following notation is used :

$\tilde{Z}$, $\tilde{Z}_3$ denote the bundles of Z, Z_3 coefficients on B which are classified by the map $w_1(\xi): B \to K(Z_2,1)$ ($K(Z_2,1)$ being the classifying space of the groups aut $Z \approx Z_2$, aut $Z_3 \approx Z_2$)

H^i denotes $H^i(B;Z_2)$

w_i is the Stiefel-Whitney class $w_i(\xi)$

p_1 is the mod 3 Pontryagin class $p_1(\xi) \in H^4(B;Z_3)$

$\mathcal{P}_3^1$ is the reduced power operation in local coefficients defined in [6].

Reduction mod 2 and mod 3 of integral classes is to be understood where required.

5.8.0 $\underline{n = 4s,\ s \geq 2}$

$$E_1^{0,q} = H^{q+n}(B;\tilde{Z}) + H^{q+n+1} + H^{q+n+3}$$

$$E_1^{1,q} = H^{q+n} + H^{q+n+1} + H^{q+n+2} + H^{q+n+2} + H^{q+n+3} + H^{q+n+2}(B;\tilde{Z}_3)$$

$$E_1^{2,q} = H^{q+n} + H^{q+n+1} + H^{q+n+1}$$

$$E_1^{3,q} = H^{q+n} \quad .$$

$$d_1^{0,*} = \begin{bmatrix} Sq^2 + w_2 & w_1 & 0 \\ 0 & Sq^2 + w_2 & 0 \\ Sq^4 + w_4 & Sq^3 & A \\ 0 & Sq^2Sq^1 + (w_1^2 + w_2)Sq^1 & Sq^1 \\ 0 & Sq^4 + w_4 & B \\ \wp_3^1 - p_1 & 0 & 0 \end{bmatrix}$$

where $A = w_1$, $B = w_2$ if $n \equiv 0 \pmod 8$

$A = Sq^1$, $B = Sq^2$ if $n \equiv 4 \pmod 8$.

$$d_1^{1,*} = \begin{bmatrix} Sq^2 + w_1Sq^1 + w_1^2 + w_2 & w_1 & 0 & 0 & 0 \\ 0 & Sq^2 + w_2 & Sq^1 & 0 & 0 \\ \begin{matrix} Sq^2Sq^1 + Sq^3 \\ + (w_1^2 + w_2)Sq^1 + w_1w_2 \end{matrix} & 0 & w_1 & 0 & 0 \end{bmatrix}$$

$$d_1^{2,*} = [Sq^2 + w_1Sq^1 + w_1^2 + w_2 \qquad w_1 \qquad Sq^1 + w_1] .$$

5.8.1 $\underline{n = 4s + 1,\ s \geq 2}$.

$$E_1^{0,q} = H^{q+n} + H^{q+n+2}$$

$$E_1^{1,q} = H^{q+n} + H^{q+n+1} + H^{q+n+2} + R$$

$$E_1^{2,q} = H^{q+n}$$

where $R = H^{q+n+4}$ if $n \equiv 1 \pmod 8$, $R = 0$ if $n \equiv 5 \pmod 8$.

$$d_1^{0,*} = \begin{bmatrix} Sq^2 + w_2 & 0 \\ Sq^2Sq^1 + (w_1^2 + w_2)Sq^1 & Sq^1 \\ Sq^4 + w_4 & A \\ 0 & B \end{bmatrix}$$

where $A = w_2$, $B = Sq^4 + w_4$ if $n \equiv 1 \pmod 8$

$A = Sq^2$, $B = 0$ if $n \equiv 5 \pmod 8$.

$$d_1^{1,*} = [Sq^2 + w_2 \quad Sq^1 \quad 0 \quad 0].$$

5.8.2 <u>$n = 4s + 2,\ s \geq 2.$</u>

$$E_1^{0,q} = H^{q+n}(B;\widetilde{Z}) + H^{q+n+1} + R$$

$$E_1^{1,q} = H^{q+n} + H^{q+n+2} + S + H^{q+n+4} + H^{q+n+2}(B;\widetilde{Z}_3)$$

$$E_1^{2,q} = H^{q+n+1} + H^{q+n+3}$$

$$E_1^{3,q} = H^{q+n+2}$$

where $R = H^{q+n+5}$, $S = H^{q+n+3}$ if $n \equiv 2 \pmod 8$

$R = S = 0$ if $n \equiv 6 \pmod 8$.

$$d_1^{0,*} = \begin{bmatrix} Sq^2 + w_2 & Sq^1 & 0 \\ Sq^4 + w_4 & A & 0 \\ 0 & B & 0 \\ 0 & \begin{matrix} Sq^4Sq^1 + (w_1^2 + w_2)Sq^3 + \\ (w_1w_2+w_3)Sq^2 + (w_2^2 + w_4)Sq^1 \end{matrix} & C \\ \wp_3^1 - p_1 & 0 & 0 \end{bmatrix}$$

where $A = w_3$, $B = Sq^4 + w_4$, $C = Sq^1$ if $n \equiv 2 \pmod 8$

$A = Sq^3 + w_1 Sq^2 + w_2 Sq^1$, $B = C = 0$ if $n \equiv 6 \pmod 8$.

$$d_1^{1,*} = \begin{bmatrix} Sq^2Sq^1 + w_1^2Sq^1 + w_3 & Sq^1 + w_1 & 0 & 0 & 0 \\ Sq^2Sq^3 + w_2Sq^2Sq^1 + (w_1w_4 + w_5) & & & & \\ \quad + (w_1w_3 + w_2^2 + w_1^2w_2)Sq^1 & 0 & 0 & Sq^1 & 0 \end{bmatrix}$$

$$d_1^{2,*} = [Sq^2Sq^1 + w_1Sq^2 + w_2Sq^1 + w_1w_2 + w_1^3 \qquad Sq^1 \ ] .$$

5.8.3. <u>$n = 4s + 3, \ s \geq 1.$</u>

$$E_1^{0,q} = H^{q+n} + R$$

$$E_1^{1,q} = H^{q+n+1} + S + H^{q+n+3} + H^{q+n+4}$$

$$E_1^{2,q} = H^{q+n+1} + H^{q+n+2}$$

$$E_1^{3,q} = H^{q+n+1}$$

where $R = H^{q+n+4}$, $S = H^{q+n+2}$ if $n \equiv 3 \pmod 8$

$R = S = 0$ if $n \equiv 7 \pmod 8$.

$$d_1^{0,*} = \begin{bmatrix} Sq^2Sq^1 + (w_1^2 + w_2)Sq^1 & 0 \\ A & 0 \\ \begin{matrix} Sq^4Sq^1 + (w_1^2 + w_2)Sq^3 + \\ (w_1w_2 + w_3)Sq^2 + (w_2^2 + w_4)Sq^1 \end{matrix} & B \\ \begin{matrix} Sq^4Sq^2 + (w_1w_2 + w_3)Sq^3 \\ + (w_2^2 + w_4)Sq^2 \end{matrix} & C \end{bmatrix}$$

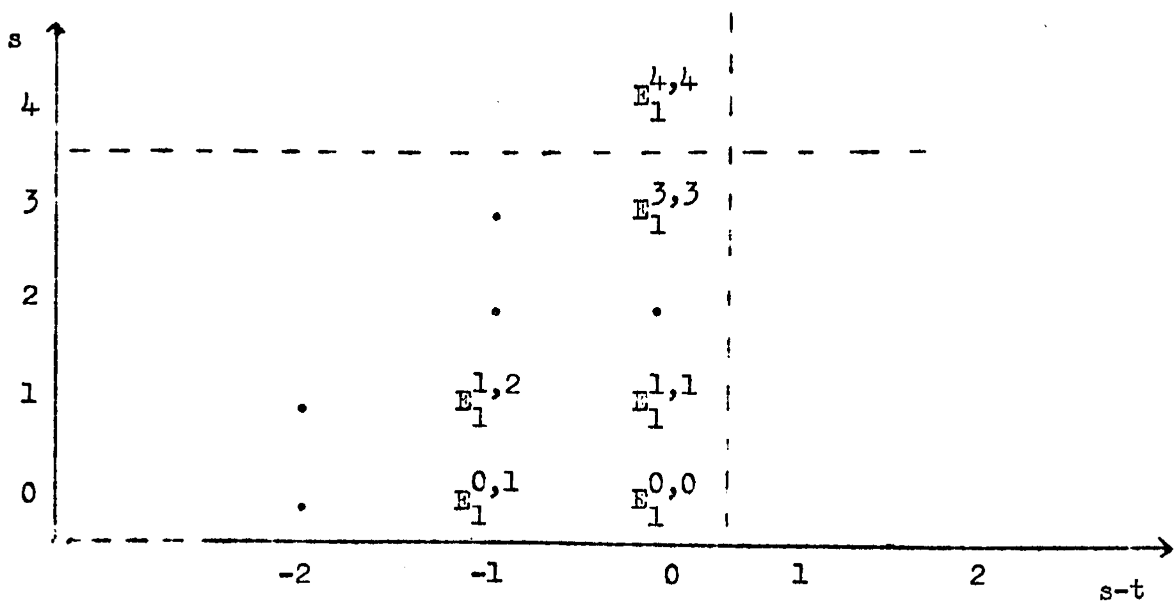
where $A = Sq^4 + w_4$, $B = Sq^1$, $C = Sq^2$ if $n \equiv 3 \pmod 8$

$A = B = C = 0$ if $n \equiv 7 \pmod 8$.

$$d_1^{1,*} = \begin{bmatrix} Sq^2 + w_1^2 + w_2 & 0 & 0 & 0 \\ Sq^2Sq^1 + w_1w_2 + w_3 & 0 & Sq^1 & 0 \end{bmatrix}$$

$$d_1^{2,*} = [Sq^2 + w_1^2 + w_2 \qquad Sq^1] .$$

5.9 We wish to use 1.9 to enumerate the cross-sections of our bundle over B with fibre $0/0_n$. Hence we need to compute the order of the group π_0^σ for this bundle. We show that if $\dim B \leq n + 5$ the above tables give enough of the spectral sequence 4.19 for us to tackle this. The spectral sequence is a half-plane one, and can be plotted as follows :



The E_1 groups below the horizontal dotted line are given in 5.8 (if $n \equiv 1 \bmod 4$, we draw the line just above $s = 2$ instead of $s = 3$). We need to find the E_∞ groups in the column $s - t = 0$. Since d_r

increases s by r and decreases $s - t$ by 1, all the differentials affecting $E_1^{0,0}, \dots, E_1^{3,3}$ have source and target either in the region below the dotted line or in the quadrant $s - t \leq 0$, $s \geq 4$ ($s \geq 3$ if $n \equiv 1 \bmod 4$). Therefore the following lemma shows we have all the groups we need provided $\dim B \leq n + 5$.

5.10 Lemma. If $\dim B \leq n + 5$, then $E_1^{s,t} = 0$ when $s - t \leq 0$ and $s \geq 4$ ($s \geq 3$ if $n \equiv 1 \bmod 4$).

Proof. The MPT's of 5.1 kill homotopy groups up to $\pi_{n+5}(O/O_n)$ in four stages (three if $n \equiv 1 \bmod 4$). By the exact sequences of 2.8, the graded coefficient module Γ^s of the SMPT has zero components in degrees $\leq n + 5$ when $s \geq 4$ ($s \geq 3$ if $n \equiv 1$). Thus for s in this range, $E_1^{s,t} = \mathcal{H}^{t-s}(B;\Gamma^s)$ is a direct sum of cohomology groups of dimensions $\geq n + 6 + t - s$, and therefore vanishes if $s - t \leq 0$ and $\dim B \leq n + 5$.

5.11 Application: enumeration of immersions in the metastable range. Let M be a smooth m-manifold. According to the theorem of Hirsch [8] as formulated by James and Thomas [10], the number of regular homotopy classes of smooth immersions of M in R^{m+n} ($n \geq 2$) is equal to the number of homotopy classes of cross-sections of the bundle with fibre O/O_n associated to the stable normal bundle of M. In the metastable range $n \geqslant \frac{1}{2}m + 1$, this bundle satisfies the hypothesis of 1.9; hence the number of immersions is either zero or the order of the group π_0^σ of the bundle. When $m \leqslant n + 5$ and $n \geqslant 7$, the spectral sequence of 5.8 can be used to compute the order of π_0^σ. If the requisite Stiefel-Whitney classes and Steenrod operations in M are known, the only remaining problem is the evaluation of the higher differentials.

We apply the method to real projective m-space P^m.

5.12 Theorem. Suppose P^m can be smoothly immersed in R^{2m-k}, where $1 \leqslant k \leqslant 5$ and $m - k \geqslant 7$. Then the number of regular homotopy classes of such immersions is as follows:

k: / m mod 8	1	2	3	4	5
0	2	1	1	1	8 or 4
1	8	4	48	1	1
2	4	2	2	1	8 or 4
3	8	32	6144	64 or 16	32
4	2	1	1	1	8 or 4
5	8	4	48	1	1
6	4	2	4	4	32 or 16
7	8	32	6144	256	256

The cases $k = 2$, $m \equiv 1 \pmod 4$ and $k = 1$, all m are covered by theorems of James and Thomas [11]. The result in the case $k = 4$, $m \equiv 1 \pmod 4$ is given by McClendon [12]. The alternatives in the last two columns are due to our failure to evaluate certain higher differentials. Every entry in the table is meaningful, for the results of [15] imply that there is an integer m in every residue class mod 8 such that P^m immerses in R^{2m-5}.

We prove 5.12 in the case $m \equiv 1 \pmod 8$, $k = 3$. The other cases are established by very similar calculations. Let ν be the stable normal bundle of P^{8x+1}, $x \geq 2$. Then $w_1(\nu), w_3(\nu), w_5(\nu)$ are zero, and $w_2(\nu), w_4(\nu)$ are non-zero. Table 5.8.2 gives the following portion of the spectral sequence for the bundle p with fibre $O/O(8x-2)$ associated with ν :

s \ $s-t$	-1	0	1	2
3	0	0	0	Z_2
2	0	$Z_2 + 0$	$Z_2 + 0$	$Z_2 + Z_2$
1	$Z_2+0+0+0$	$Z_2+Z_2+0+Z_3$	Z_2+Z_2+0+0	$Z_2+Z_2+Z_2+0$
0	$0 + Z_2$	$Z_2 + Z_2$	$0 + Z_2$	$Z_2 + Z_2$

The non-zero components of $d_1^{s,t}$ are represented by solid arrows. There is only one higher differential in the diagram which could be non-zero, namely $d_2^{0,-1} : H^{8x-2}(P^{8x+1}; Z_2) \to H^{8x+1}(P^{8x+1}; Z_2)$ (the dotted arrow). We show $d_2^{0,-1}$ is zero. Since ν is an even multiple of the Hopf line bundle over P^{8x+1}, ν is induced from a stable real vector bundle ξ over the complex projective space CP^{4x} by the natural projection $\pi : P^{8x+1} \to CP^{4x}$ [1]. Let p' be the bundle with fibre $O/O(8x-2)$ associated with ξ, and denote by $'E_r^{**}$, $'d_r^{**}$ its spectral sequence. By naturality there is a commutative diagram

$$\begin{tikzcd}
{'E_2^{2,2}} \arrow[r, "\pi^*"] & E_2^{2,2} \\
{'E_2^{0,-1}} \arrow[u, "{'d_2^{0,-1}}"] \arrow[r, "\pi^*"] & E_2^{0,-1} \arrow[u, "d_2^{0,-1}"']
\end{tikzcd}$$

But $\pi^* : {'E_2^{0,-1}} \to E_2^{0,-1}$ is $\pi^* : H^{8x-2}(CP^{4x}; Z_2) \to H^{8x-2}(P^{8x+1}; Z_2)$, so is an isomorphism : and $'E_2^{2,2}$ is zero from 5.8.2. By commutativity, $d_2^{0,-1} = 0$.

Hence $E_\infty^{0,0} \approx Z_2$, $E_\infty^{1,1} \approx Z_2 + Z_2 + Z_3$, $E_\infty^{2,2} \approx Z_2$,

so $\pi_0^\sigma(p)$ has order 48. By 5.11 the number of immersions of P^{8x+1} in R^{16x-1} is either 48 or 0. This gives one entry in table 5.12. In fact P^{8x+1} does immerse in R^{16x-1} [20]: it should be possible to recover this result from the spectral sequence and 1.11 by observing $\pi_{-1}^\sigma(p) \approx E_\infty^{0,1} \approx E_1^{0,1} \approx H^{8x}(P^{8x+1};Z_2)$ and showing that under this isomorphism the obstruction element $\mathfrak{O}(p)$ of 1.11 is mapped to $w_{8x}(\nu) = 0$.

Theorem 5.12 has some geometrically interesting corollaries.

5.13 Corollary. Let f be a smooth immersion of P^m in R^{2m-2} or R^{2m-3}, where $m = 4s \geq 12$ and m is not a power of 2. Then f is regularly homotopic to an embedding.

Proof. There is an embedding of P^m in R^{2m-3} under these conditions by [13]. According to 5.12 this embedding belongs to the same regular homotopy class as f.

§6. Enumerating unstable vector bundles

6.0 Let X be a connected n-dimensional complex, and ξ a stable isomorphism class of real vector bundles over X. Then ξ can be realised by an $(n+1)$-plane bundle over X, and this bundle is unique. The question arises: given $k < n$, how many k-plane bundles belong to the class ξ (if any)? James and Thomas [9] give a formula for the answer in the case $k = n$, provided either n is odd or $w_1(\xi) = 0$. From this they compute the number of n-dimensional normal bundles to P^n. We obtain further results by generalising the method of [9]. We show (6.13) that a formula analogous with those of [9] gives the number of n-plane bundles in the class ξ also for n even and $w_1(\xi) \neq 0$. Then we attempt to generalise the method to tackle the problem when $k < n$: we obtain no neat formulae, but we compute the number of $(n-1)$- and $(n-2)$-dimensional normal bundles to P^n in certain cases (6.16).

We now make explicit the equivalence relation modulo which we intend to enumerate bundles. Throughout §6 we adopt the following terminology: a k-plane bundle over X means a k-dimensional real vector bundle together with a preferred orientation for the fibre over the base-point of X. Isomorphisms of k-plane bundles are required to preserve the orientation on the fibre. (In the case of orientable vector bundles, an orientation on the fibre determines an orientation on the bundle, and we are classifying bundles up to oriented isomorphism.) The isomorphism classes of k-plane bundles over X are therefore in one-one correspondence with the <u>based</u> homotopy classes $X \to BO_k$.

6.1 Let the stable class ξ be classified by $f : X \to BO$. Regard the natural map $p : BO_k \to BO$ as the projection of the universal bundle with fibre O/O_k. The k-plane bundles in the stable class ξ are in one-one correspondence with the elements of the inverse image $p_*^{-1}\{f\}$ of the homotopy class of f by $p_* : [X,BO_k] \to [X,BO]$: here $[X,Y]$ is the set of based homotopy classes $X \to Y$, and we shall denote the base-point of any space by o.

Consider a more general situation. Let $q : Y \to BO$ be any fibration: we investigate $q_*^{-1}\{f\} \subset [X,Y]$. Any element of $q_*^{-1}\{f\}$ has a representative $g : X \to Y$ such that $qg = f : X \to BO$ (by the covering homotopy property). We call g a lifting of f. Let $g_0, g_1 : X \to Y$ be liftings of f : we say g_0, g_1 are <u>homotopic over f</u> if they are connected by a homotopy $g_t : X \to Y$ such that $qg_t = f$ for all $t \in I$. As in 2.2, we denote by $[X,Y]_f$ the set of homotopy classes over f of liftings of f.

Then

(A) there is a canonical surjection $[X,Y]_f \to q_*^{-1}\{f\}$

(B) $[X,Y]_f$ is canonically isomorphic to the set of homotopy classes of sections of the fibration induced from $q : Y \to BO$ by $f: X \to Y$.

When q is $p : BO_k \to BO$, (B) shows that $[X,BO_k]_f$ is equivalent to the set of sections of the O/O_k-bundle associated with the stable class ξ. Hence we can hope to calculate its order from the spectral sequences 5.8. We then investigate the surjection (A) by a method now to be described, the idea of which is due to James and Thomas [9].

We define an action of the group $\widetilde{KO}^{-1}X$ on $[X,Y]_f$. Consider the diagram

$$SX \xleftarrow{m} X \times S^1 \xrightarrow{\pi} X$$

where π is the projection and m the identification map $X \times S^1 \to X_\wedge S^1 = SX$ (we regard S^1 as $I/\partial I$). An element $\beta \in \widetilde{KO}^{-1}X$ is a stable bundle over SX. From β and our stable bundle ξ we obtain a stable bundle over $X \times S^1$.

6.2 <u>Definition</u> $\langle\beta,\xi\rangle$ is the Whitney sum $m^*\beta \oplus \pi^*\xi$.

Then $\langle\beta,\xi\rangle | X \times o$ is stably isomorphic to ξ, and $\langle\beta,\xi\rangle | o \times S^1$ is trivial. Hence $\langle\beta,\xi\rangle$ can be classified by a map $F = F_\beta : X \times S^1 \to BO$ such that $F(x,0) = fx$, $F(o \times S^1) = o$. Then the composite $X \times I \to X \times S^1 \xrightarrow{F} BO$ of F_β with the identification map is a based self-homotopy of $f : X \to BO$. For any element $\eta \in [X,Y]_f$, this homotopy can be lifted to a homotopy $G : X \times I \to Y$ such that $G|X \times o$ represents η.

6.3 Definition $\beta\cdot\eta \in [X,Y]_f$ is the class of $G|X \times 1$.

It is easy to show $\beta\cdot\eta$ is well-defined. To show that 6.3 gives an action of $\widetilde{KO}^{-1}X$ on $[X,Y]_f$, we must verify that $(\beta_1 + \beta_2)\cdot\eta = \beta_1\cdot(\beta_2\cdot\eta)$ for $\beta_1,\beta_2 \in \widetilde{KO}^{-1}X$. This follows from the fact that addition in $\widetilde{KO}^{-1}X \approx [SX,BO]$ is induced by the H'-structure $SX \to SX \vee SX$, whence the self-homotopy of f obtained from $F_{\beta_1+\beta_2} : X \times S^1 \to BO$ can be taken to be the track sum of those obtained from F_{β_1}, F_{β_2}.

We can describe the surjection 6.1 (A) in terms of this action.

6.4 Proposition (i) In the situation of 6.1, $q_*^{-1}\{f\}$ is the orbit set of the action 6.3. Explicitly, $\eta_o,\eta_1 \in [X,Y]_f$ have the same image in $[X,Y]$ iff there exists $\beta \in \widetilde{KO}^{-1}X$ such that $\beta\cdot\eta_o = \eta_1$.

(ii) Let $\beta \in \widetilde{KO}^{-1}X$, $\eta \in [X,Y]_f$. Then $\beta\cdot\eta = \eta$ iff there is a classifying map $X \times S^1 \to BO$ for $\langle\beta,\xi\rangle$ which has a lifting $H : X \times S^1 \to Y$ such that $H|X \times o$ represents η.

Proof. (i) If $\beta\cdot\eta_o = \eta_1$, then by construction η_o and η_1 have representatives which are (based) homotopic as maps $X \to Y$. Conversely, suppose η_o and η_1 have homotopic representatives g_o, g_1. Take a homotopy $G : g_o \simeq g_1$. Since g_o, g_1 lift f, $qG : X \times I \to BO$ is a (based) self-homotopy of $f : X \to BO$. So qG factors through a map $X \times S^1 \to BO$, which represents a stable class $\phi \in \widetilde{KO}(X \times S')$ whose restriction to $\widetilde{KO}(X \vee S^1) \approx \widetilde{KO}\, X \oplus \widetilde{KO}\, S^1$ is $(\xi,0)$. Let π, m be as in the diagram preceding 6.2. The element $\phi - \pi^*\xi \in \widetilde{KO}(X \times S^1)$ restricts to zero in $\widetilde{KO}(X \vee S^1)$, so $\phi - \pi^*\xi = m^*\beta$ for some $\beta \in \widetilde{KO}(SX)$. By definition of the action, $\beta\cdot\eta_o = \eta_1$.

(ii) Let $F : X \times S^1 \to BO$ be a classifying map for $\langle\beta,\xi\rangle$ such that $F(x,0) = fx$. Then $\beta\cdot\eta = \eta$ iff the homotopy $X \times I \to X \times S^1 \xrightarrow{F} BO$ can be lifted to a homotopy $G : X \times I \to Y$ such that $G|X \times 0$, $G|X \times 1$ represent η. If this happens, we can arrange $G(x,0) = G(x,1)$ by the covering homotopy property: hence $\beta\cdot\eta = \eta$ iff F lifts to $H : X \times S^1 \to Y$ with $H|X \times o$ representing η.

6.5 In the applications of 6.4, we need some results about classical obstruction and difference classes. Let $p : Y \to B$ be a fibration, and let

$$\begin{tikzcd} & & \mathcal{K}(\Gamma^{m-1},1) & & \mathcal{K}(\Gamma^{o},1) \\ \mathcal{M}: & Y \arrow[r] & Y^{m} \arrow[r] & Y^{m-1} \arrow[u, "k^{m-1}"] \arrow[r] & \cdots \arrow[r] & Y^{o} \arrow[u, "k^{o}"] = B \end{tikzcd}$$

be an MPT for p. Suppose a map $f : X \to Y^r$ is given, where $r < m$ and X is a complex. Since $Y^{r+1} \to Y^r$ is induced from the universal fibration over $\mathcal{K}(\Gamma^r,1)$ by $k^r : Y^r \to \mathcal{K}(\Gamma^r,1)$, there is a lifting $g : X \to Y^{r+1}$ iff $f^*k^r \in \mathcal{H}^1(X;\Gamma^r)$ is zero. If liftings to Y^{r+1} exist, they are enumerated (up to homotopy over f) by difference classes in $\mathcal{H}^o(X;\Gamma^r)$. Explicitly, if $g,h : X \to Y^{r+1}$ lift f, then an element $D(h,g) \in \mathcal{H}^o(X;\Gamma^r)$ is defined; and as h varies, the correspondence $h \to D(h,g)$ is a bijection from $[X,Y^{r+1}]_f$ onto $\mathcal{H}^o(X;\Gamma^r)$. To prove this, one notes that $[X,Y^{r+1}]_f$ is isomorphic to the set of homotopy classes of sections of the fibration induced by f from $Y^{r+1} \to Y^r$, and the result follows from application of 1.8 and 4.19. An alternative proof follows from the following reinterpretation of $D(h,g)$, which will be needed later. Let $\phi : Z \to Y^{r+1}$ be the fibration induced from $Y^{r+1} \to Y^r$ by $Y^{r+1} \to Y^r$. Then there is a diagram containing two pullback squares:

$$
\begin{tikzcd}
Z \arrow[r] \arrow[d, "\phi"'] & Y^{r+1} \arrow[r] \arrow[d] & P_B\mathcal{K}(\Gamma^r,1) \arrow[d] \\
Y^{r+1} \arrow[r] \arrow[ur, dashed] & Y^r \arrow[r, "k^r"] \arrow[d] & \mathcal{K}(\Gamma^r,1) \\
& \vdots \arrow[d] & \\
& B &
\end{tikzcd}
$$

The horizontal map $Y^{r+1} \to \mathcal{K}(\Gamma^r,1)$ has a standard factorisation through $P_B\mathcal{K}(\Gamma^r,1)$ via the dotted map $1 : Y^{r+1} \to Y^{r+1}$. Hence there is a standard homotopy from $Y^{r+1} \to \mathcal{K}(\Gamma^r,1)$ to the composite $Y^{r+1} \to B \xrightarrow{s} \mathcal{K}(\Gamma^r,1)$ where s is the inclusion of the zero section in the Eilenberg-MacLane object $\mathcal{K}(\Gamma^r,1)$. Hence, by the diagram in 2.6, $\phi : Z \to Y^{r+1}$ is fibre homotopy equivalent in a standard way to the induced Eilenberg-MacLane object $\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1} \to Y^{r+1}$ over Y^{r+1}. By abuse of language, we may regard ϕ (for homotopy purposes) as an Eilenberg-MacLane object, and we have a pullback diagram:

$$
\begin{tikzcd}
\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1} \arrow[r, "\mu"] \arrow[d, shift left, "\phi"] & Y^{r+1} \arrow[d] \\
Y^{r+1} \arrow[r] \arrow[u, shift left, "s'"] & Y^r .
\end{tikzcd}
\tag{6.6}
$$

Here s' is the zero section, and by construction $\mu s' \simeq 1$. Now let $g : X \to Y^{r+1}$ be any lift of $f : X \to Y^r$. Then maps $h : X \to Y^{r+1}$ lifting f correspond to maps $\bar{h} : X \to \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}$ with $\phi\bar{h} = g$:

(6.7)
$$\begin{tikzcd}
X \arrow[dr, "\bar{h}"] \arrow[drr, bend left, "h"] \arrow[ddr, "g"'] & & \\
& \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1} \arrow[r, "\mu"'] \arrow[d, shift left, "\phi"] & Y^{r+1} \arrow[d] \\
& Y^{r+1} \arrow[u, shift left, "s'"] \arrow[r] & Y^r
\end{tikzcd}$$

Hence $\{h\} \to \{\bar{h}\}$ is a bijection

$$[X, Y^{r+1}]_f \to [X, \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}]_g.$$

6.8 <u>Lemma</u> Let ι be the universal class of the Eilenberg-MacLane object ϕ. Then in the notation of the diagram, $D(h,g) = \bar{h}^*\iota$.

Proof. By naturality, $D(h,g) = D(\bar{h},\bar{g})$ (where $\bar{g}$ satisfies $\mu\bar{g} = \phi\bar{g} = g$). But $D(\bar{h},\bar{g}) = \bar{h}^*\iota - \bar{g}^*\iota$ because the representation isomorphism $\mathcal{H}^0(X;\Gamma^r) \approx [X, \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}]_g$ is defined by difference classes (2.3,2.6). Since $\mu s' \simeq 1$, we have $gs' \simeq \bar{g}$, so $\bar{g}^*\iota = 0$. Therefore $D(h,g) = \bar{h}^*\iota$.

If $g,h : X \to Y^{r+1}$ lift f, then we have obstructions g^*k^{r+1}, h^*k^{r+1} to lifting g,h respectively to Y^{r+2}.

6.9 Proposition. Suppose the MPT $\mathcal{U}$ of 6.5 is the first tower $\mathcal{U}_0$ of an SMPT for $p : Y \to B$. Then $h^*k^{r+1} - g^*k^{r+1} = d_1^{r,r}D(h,g)$, where $d_1^{r,r}$ is the differential in the spectral sequence 4.19.

Proof. Using the natural map $Y \to Y^{r+1}$, we add one more induced fibration ψ to diagram 6.6:

$$\begin{tikzcd}
\mathcal{K}(\Gamma^r,0)\times_B Y \arrow[r, "\rho"] \arrow[d, "\psi"'] & \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1} \arrow[r, "\mu"] \arrow[d, shift left, "\phi"] & Y^{r+1} \arrow[r, "k^{r+1}"] \arrow[d] & \mathcal{K}(\Gamma^{r+1},1) \\
Y \arrow[r] & Y^{r+1} \arrow[r] \arrow[u, shift left, "s'"] & Y^r \arrow[r, "k^r"] & \mathcal{K}(\Gamma^r,1).
\end{tikzcd}$$

The composite $\mu\rho$ is homotopic to the map ν of 4.17 (with Y in place of X): for the fibration ϕ is identified with an Eilenberg-MacLane fibration by the canonical homotopy of $Y^{r+1} \to \mathcal{K}(\Gamma^r,1)$ into the zero section, so ψ is identified with $\mathcal{K}(\Gamma^r,0)\times_B Y \to Y$ by the composite of this homotopy with $Y \to Y^{r+1}$, and this is the method of 4.2.

Let $\bar{h} : X \to \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}$ satisfy $\phi\bar{h} = g$, $\mu\bar{h} = h$ (diagram 6.7). Then $h^*k^{r+1} = \bar{h}^*\mu^*k^{r+1}$. We determine $\mu^*k^{r+1} \in \mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1};\Gamma^{r+1})$. The projection $\phi : \mathcal{K}(\Gamma^r,0)\times_B Y^{r+1} \to Y^{r+1}$ splits the cohomology sequence of the pair $(\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}, Y^{r+1})$. Hence we have a split short exact sequence

$$\mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1},Y^{r+1};\Gamma^{r+1}) \xrightarrow{j^*} \mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}) \underset{\phi^*}{\overset{s'^*}{\rightleftarrows}} \mathcal{H}^1(Y^{r+1};\Gamma^{r+1})$$

and so $\mu^*k^{r+1} = \phi^*s'^*\mu^*k^{r+1} + j^*\alpha$ for some unique $\alpha \in \mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1},Y^{r+1};\Gamma^{r+1})$. Since $s'\mu \simeq 1$, $\mu^*k^{r+1} = \phi^*k^{r+1} + j^*\alpha$. Consider the images of these classes under the homomorphism of cohomology sequences induced by the map $\rho : (\mathcal{K}(\Gamma^r,0)\times_B Y,Y) \to (\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1},Y^{r+1})$. Since $\mu\rho \simeq \nu$, we can apply 4.18 to deduce that $\rho^*\mu^*k^{r+1}$ lies in $\mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y,Y;\Gamma^{r+1})$ and is the universal example for the twisted operation $d_1^{r,r}$ in the spectral sequence of the induced fibration $Y \times_B Y \to Y$. Hence $\rho^*\phi^*k^{r+1} = 0$, and $\rho^*\alpha$ is this universal example. We claim

$$\rho^* : \mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1},Y^{r+1};\Gamma^{r+1}) \to \mathcal{H}^1(\mathcal{K}(\Gamma^r,0)\times_B Y,Y;\Gamma^{r+1})$$

is an isomorphism. For if the fibre of $p : Y \to B$ has connectivity c, then $\rho : (\mathcal{K}(\Gamma^r,0)\times_B Y,Y) \to (\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1},Y^{r+1})$ is a fibration-pair whose fibre and base-pair both have connectivity $\geqslant c$; and Γ^{r+1} has zero groups in gradings $> 2c$ by definition of SMPT 3.4 (ii). So ρ^* is iso by the Leray-Serre spectral sequence. Hence, by naturality of the spectral sequence 4.19, α is the universal example for the operation $d_1^{r,r}$ in the fibration over Y^{r+1}. We have

$$h^*k^{r+1} = \bar{h}^*\mu^*k^{r+1} = \bar{h}^*\phi^*k^{r+1} + \bar{h}^*\alpha$$
$$= g^*k^{r+1} + \bar{h}^*d_1^{r,r}(\iota)$$

where ι is the universal class of $\mathcal{K}(\Gamma^r,0)\times_B Y^{r+1}$. By 6.8

$$\bar{h}^*d_1^{r,r}(\iota) = d_1^{r,r}\bar{h}^*\iota = d_1^{r,r}D(h,g)$$

and so $h^*k^{r+1} - g^*k^{r+1} = d_1^{r,r}D(h,g)$.

6.10 <u>The primary obstruction case</u>. Suppose dim $X = k$, where k is even, and we wish to find the number of k-plane bundles in a stable class ξ classified by $f : X \to BO$. We construct an MPT of height 1 for $p : BO_k \to BO$ by taking $W_{k+1} \in H^{k+1}(BO;\tilde{Z})$ as k-invariant:

$$\begin{array}{ccccc} & & BO_k & & \\ & & \downarrow & & \\ & & Y^1 & & \\ & & \downarrow p_1 & & \\ X & \xrightarrow{f} & BO & \xrightarrow{W_{k+1}} & K_{BO}(\tilde{Z},k+1) \end{array}$$

This MPT kills $\pi_k(O/O_k) \approx Z$: hence the fibre of $BO_k \to Y^1$ is k-connected, so $[X,BO_k] \approx [X,Y^1]$ and the number of k-plane bundles is the order of $p_{1*}^{-1}\{f\} \subset [X,Y^1]$. By 6.4(i), $p_{1*}^{-1}\{f\}$ is the orbit set of $[X,Y^1]_f$ under $\widetilde{KO}^{-1}X$. The next proposition determines the action.

Let $g : X \to Y^1$ be any lifting of f (these certainly exist) and $\{g\}$ its class in $[X,Y^1]_f$. If $\beta \in \widetilde{KO}^{-1}X$, the element $\beta\cdot\{g\} \in [X,Y^1]_f$ is completely determined by the difference $D(\beta\cdot\{g\}, \{g\}) \in H^k(X; \tilde{Z})$, where $\tilde{Z}$ is the system of integer coefficients twisted by $w_1(\xi)$: $X \to K(Z_2,1) = K\ (\mathrm{aut}Z,1)$.

6.11 Proposition $D(\beta\cdot\{g\}, \{g\}) = \delta \sum_{i=2}^{k} \sigma w_i \beta.\ w_{k-i}\xi$ where $\sigma : H^i(SX;\ Z_2) \to H^{i-1}(X;Z_2)$ is the suspension isomorphism and $\delta : H^{k-1}(X;Z_2) \to H^k(X;\tilde{Z})$ is the Bockstein operation associated with mod 2 reduction of the twisted integer coefficients $\tilde{Z}$.

Proof. Let $\langle\beta,\xi\rangle$ be the stable bundle of 6.2; let $F : X \times S^1 \to BO$ be a classifying map for $\langle\beta,\xi\rangle$ such that $F(x,0) = fx$; and let $G : X \times I \to Y^1$ be a lifting of $X \times I \to X \times S^1 \xrightarrow{F} BO$ such that $G(x,0) = gx$. Then the map $h : X \to Y^1$ defined by $hx = G(x,1)$ represents $\beta\cdot\{g\}$. Consider the commutative diagram

(6.12)
$$\begin{tikzcd}
X \times \partial I \arrow[rr, phantom, "\subset"] \arrow[ddd, "\mathrm{proj}"'] \arrow[dr, dashed, "(\bar{g},\bar{h})"] & & X \times I \arrow[d, "G"] \\
& K_{BO}(\tilde{Z},k)\times_{BO}Y^1 \arrow[r, "\mu"] \arrow[dd, "\phi"] \arrow[ddr, phantom, "\text{A}"] & Y^1 \arrow[dd] \\
& & \\
X \arrow[r, "g"'] & Y^1 \arrow[r] & BO
\end{tikzcd}$$

where (A) is a case of 6.6. The dotted map exists by the pullback property, and consists of maps

$\bar{g} : X \times 0 \to K_{BO}(\tilde{Z},k)\times_{BO}Y^1$, $\bar{h} : X \times 1 \to K_{BO}(\tilde{Z},k)\times_{BO}Y^1$.

By 6.8, $\bar{h}^*\iota = D(\beta\cdot\{g\},\{g\})$, $\bar{g}^*\iota = 0$ where ι is the universal class. Consider the composite

$$H^k(K_{BO}(\tilde{Z},k)\times_{BO}Y^1;\ \tilde{Z}) \xrightarrow{\tau} H^{k+1}(BO,Y^1;\tilde{Z}) \xrightarrow{j^*} H^{k+1}(BO;\tilde{Z})$$ where

τ is the relative transgression in the induced fibration square (A), $H^{k+1}(BO,Y^1;\tilde{Z})$ is defined in terms of the mapping cylinder of $Y^1 \to BO$, and j^* is induced by inclusion. $j^*\tau$ maps ι to $-W_{k+1}$, as follows from the commutative diagram

$$\begin{array}{ccc} H^k(K_{BO}(\widetilde{Z},k)\times_{BO}Y^1;\widetilde{Z}) & \xrightarrow{\tau} & H^{k+1}(BO,Y^1;\widetilde{Z}) \\ i^* \downarrow & & \downarrow j^* \\ H^k(K(Z,k);Z) & \xrightarrow{t} & H^{k+1}(BO;\widetilde{Z}) \end{array}$$

in which $i : K(Z,k) \subset K_{BO}(\widetilde{Z},k)\times_{BO}Y^1$ is the inclusion of the fibre in the Eilenberg-MacLane fibration, and t is the transgression in the fibration $K(Z,k) \subset Y^1 \to BO$.

Diagram 6.12 yields a commutative diagram

$$\begin{array}{ccc} H^k(K_{BO}(\widetilde{Z},k)\times_{BO}Y^1;\widetilde{Z}) & \xrightarrow{(\tilde{g},\tilde{h})^*} & H^k(X\times \partial I;\widetilde{Z}) \\ \delta \downarrow & & \downarrow \delta \\ H^{k+1}(Y^1,K_{BO}(\widetilde{Z},k)\times_{BO}Y^1;\widetilde{Z}) & \xrightarrow{G^*} & H^{k+1}(X \times I, X \times \partial I;\widetilde{Z}) \\ p_1^* \uparrow & & \uparrow \approx \\ H^{k+1}(BO,Y^1;\widetilde{Z}) & \xrightarrow{\Psi} & H^{k+1}(X \times S^1, X;\widetilde{Z}) \\ j^* \downarrow & & \downarrow \approx \\ H^{k+1}(BO;\widetilde{Z}) & \xrightarrow{F^*} & H^{k+1}(X \times S^1;\widetilde{Z}) \end{array}$$

where Ψ is induced by $F : X \times S^1 \to BO$ and $g : X \to Y^1$. Since $j^*\tau\iota = -W_{k+1}$, there is a class $\gamma \in H^{k+1}(BO, Y^1; \tilde{Z})$ with $p_1^*\gamma = \delta\iota$ and $j^*\gamma = -W_{k+1}$. Now

$(\bar{g}, \bar{h})^*\iota = (0, D(\beta \cdot \{g\}, \{g\}) \in H^k(X \times \partial I; \tilde{Z}) \approx H^k(X \times 0) \oplus H^k(X \times 1)$ as proved above, and $\delta(0, D(\beta \cdot \{g\}, \{g\})$ is

$D(\beta \cdot \{g\}, \{g\}) \times \bar{\omega} \in H^{k+1}(X \times I, X \times \partial I; \tilde{Z})$ where $\bar{\omega}$ generates $H^1(I, \partial I; Z)$. By commutativity, $\Psi\gamma \in H^{k+1}(X \times S^1, X; \tilde{Z})$ maps to $D(\beta \cdot \{g\}, \{g\}) \times \bar{\omega} \in H^{k+1}(X \times I, X \times \partial I; \tilde{Z})'$ and to $-F^* W_{k+1} = -W_{k+1}(\langle \beta, \xi \rangle)$ in $H^{k+1}(X \times S^1; \tilde{Z})$. Hence $W_{k+1}\langle \beta, \xi \rangle = D(\beta \cdot \{g\}, \{g\}) \times \omega$ where ω generates $H^1(S^1; Z)$. But $W_{k+1}\langle \beta, \xi \rangle = \tilde{\delta} w_k \langle \beta, \xi \rangle$ where $\tilde{\delta}$ is the twisted Bockstein ([22], §38.8), and $w_k\langle \beta, \xi \rangle = w_k(m^*\beta \oplus \pi^*\xi)$ (6.2)

$$= \sum_{i=0}^{k} m^* w_i \beta \cdot \pi^* w_{k-i} \xi$$

$$= w_k \xi \otimes 1 + \sum_{i=2}^{k} (\sigma w_i \beta \otimes \omega) \cdot (w_{k-i} \xi \otimes 1)$$

(for $w_1\beta$ lies in a zero group)

$$= w_k \xi \otimes 1 + \left(\sum_{i=2}^{k} \sigma w_i \beta \cdot w_{k-i} \xi\right) \otimes \omega$$

$$\therefore D(\beta \cdot \{g\}, \{g\}) \times \omega = \tilde{\delta} w_k \langle \beta, \xi \rangle$$

$$= \left(\tilde{\delta} \sum_{i=2}^{k} \sigma w_i \beta \cdot w_{k-i} \xi\right) \times \omega$$

since $\tilde{\delta} w_k \xi = 0$ for dimensional reasons and ω is an integral class. 6.11 follows.

The following generalises 1.8 of [9].

6.13 Theorem. Let ξ be a stable real vector bundle over a k-dimensional complex X, where k is even. Then the number of k-plane bundles (modulo isomorphism preserving fibre-orientation) stably equivalent to ξ equals the order of the cokernel of the homomorphism

$$\Delta : \widetilde{KO}^{-1}X \to H^k(X;\tilde{Z})$$

defined by $\Delta(\beta) = \tilde{\delta} \sum_{i=2}^{k} \sigma w_i \beta \cdot w_{k-i} \xi$, where $\tilde{Z}$, σ, $\tilde{\delta}$ are as in 6.11.

Proof. Δ is a homomorphism because $w_i(\beta_1 \oplus \beta_2) = w_i\beta_1 + w_i\beta_2$ for $\beta_1, \beta_2 \in \widetilde{KO}(SX)$. In the notation of 6.10 , the elements of $[X, Y^1]_f$ are enumerated by difference classes in $H^k(X;\tilde{Z})$; and 6.11 shows that two elements lie in the same $\widetilde{KO}^{-1}X$-orbit iff their difference class lies in the image of Δ. Hence the orbits are in (non-canonical) one-one correspondence with coker Δ.

6.14 <u>(n-1)- and (n-2)-dimensional normal bundles to P^n</u>.

In principle, one can use 6.4(i) to determine the number of k-plane bundles in a stable class ξ over X when $\dim X > k$ (taking $Y = BO_k$, $q = p : BO_k \to BO$ in 6.4). But the action of $\widetilde{KO}^{-1}X$ on $[X, BO_k]_f$ seems difficult to compute in general. We restrict our attention to the case when X is real projective n-space P^n, $n \not\equiv 3 \pmod 4$, and ξ is the stable normal bundle. If $k \geq 7$, the order of $[P^n, BO_k]_f$ is given by 5.12. By the results of [19], $\widetilde{KO}^{-1}(P^n) \approx Z_2$ for $n \not\equiv 3$ (mod 4). This greatly simplifies the calculation.

The following lemma is immediate from 1.9 of [9].

6.15 <u>Lemma</u>. Let $\sigma : H^i(SP^n;Z_2) \to H^{i-1}(P^n;Z_2)$ be the suspension isomorphism, and x the generator of $H^1(P^n;Z_2)$. Then if $n \not\equiv 3 \pmod 4$, the Stiefel-Whitney classes of the non-trivial element $\beta \in \widetilde{KO}^{-1}(P^n)$ satisfy $\sigma w_i\beta = x^{i-1}$ $(1 < i \leq n+1)$.

6.16 <u>Theorem</u>. Let $n \not\equiv 3 \pmod 4$; let k be either $n-1$ or $n-2$, $k \geq 7$. If $k = n-2$, assume P^n can be immersed in R^{2n-2}. Then the number of k-dimensional normal bundles to P^n

(modulo isomorphism preserving fibre-orientation) is as follows

n mod 4:	0	1	2
$k = n-1$	$\begin{cases} 1 & \text{if } n = 2^q \\ 2 & \text{otherwise} \end{cases}$	6	3
$k = n-2$	1	3	1 or 2

Proof. As before, let $p : BO_k \to BO$ be the standard fibration, and let $f : P^n \to BO$ classify the stable normal bundle ξ.

(a) <u>$n = 4s$, $k = n-1$</u>. This is proved in [9], 1.12.

(b) <u>$n = 4s$, $k = n-2$.</u> By 5.12, there is exactly one section of the O/O_k-bundle associated with ξ. Hence (6.1) $[P^n, BO_k]_f$ contains one element, and there is exactly one k-plane bundle in the class ξ.

(c) <u>$n = 4s+1$, $k = n-1$</u>. From the MPT 5.1 for $BO_{n-1} \to BO$ we obtain the following MPT killing $\pi_{n-1}(O/O_{n-1}) \approx Z$ and $\pi_n(O/O_{n-1}) \approx Z_2 + Z_2$:

$$\begin{tikzcd}
 & BO_{n-1} \arrow[d] & \\
 & Y^2 \arrow[d] & \\
 & Y^1 \arrow[d] \arrow[r, "k^1"] & K(Z_2,n+1) \\
P^n \arrow[r, "f"] & BO \arrow[r, "W_n \times w_{n+1}"] & K_{BO}(\tilde{Z},n) \times K(Z_2,n+1)
\end{tikzcd}$$

and by 5.8.0 the differential $d_1^{o,o} : H^{n-1}(B;\tilde{Z}) + H^n(B;Z_2) \to H^{n+1}(B;Z_2)$ in the induced spectral sequence for a stable bundle θ over base B is given by $(a,b) \to Sq^2 a + w_2(\theta)\cdot a + w_1(\theta)\cdot b$.

$f : P^n \to BO$ lifts to Y^1, since $W_n\xi = 0$, $w_{n+1}\xi = 0$. The liftings are enumerated (up to homotopy over f) by difference classes in $H^{n-1}(P^n;Z) + H^n(P^n;Z_2)$, so $[P^n,Y^1]_f$ has four elements. Represent these by maps $f_1, \ldots, f_4 : P^n \to Y^1$ such that $D(f_1,f_2)$ generates $H^n(P^n;Z_2)$ and $D(f_1,f_3)$ generates $H^{n-1}(P^n;Z)$. Each f_i lifts to Y^2 (the obstruction is in a zero group), and the liftings of each are enumerated by $H^n(P^n;Z_2)$. Let f_{i1},f_{i2} represent the two liftings of f_i. Since $[P^n,BO_k]_f \approx [P^n,Y^2]_f$ has eight elements (5.12), the eight f_{ij} belong to distinct elements of $[P^n,Y^2]_f$.

Let $\beta \in \widetilde{KO}^{-1}(P^n)$ be the non-trivial element, and let $F : P^n \times S^1 \to BO$ be a classifying map for the stable bundle $\langle \beta, \xi \rangle$ of 6.2 such that $F(x,o) = fx$, $x \in P^n$. For each pair (i,j) with $1 \leq i \leq 4$, $j = 1$ or 2 we attempt to lift F to $F_{ij} : P^n \times S^1 \to Y^2$ such that $F_{ij}(x,o) = f_{ij}x$.

The known Stiefel-Whitney classes of ξ and β (6.15) quickly yield $W_n \langle \beta, \xi \rangle = 0$, $w_{n+1} \langle \beta, \xi \rangle = 0$. Hence F lifts to $P^n \times S^1 \to Y^1$. The liftings are enumerated by $H^{n-1}(P^n \times S^1; Z) + H^n(P^n \times S^1; Z_2)$. Since $H^*(P^n \times S^1) \to H^*(P^n \times o)$ is an epimorphism, there exist liftings $F_i : P^n \times S^1 \to Y^1$ $(i = 1, \ldots, 4)$ such that $F_i(x,o) = f_i x$. F_i lifts to Y^2 iff the obstruction $F_i^* k^1 \in H^{n+1}(P^n \times S^1; Z_2)$ vanishes. Now F_i is not uniquely defined up to homotopy over F, but any two choices for F_i agree on $P^n \times o$, so their difference class lies in $H^{n-1}(P^n \times S^1, P^n \times o; Z) + H^n(P^n \times S^1, P^n \times o; Z_2)$. Since this group is annihilated by $d_1^{o,o}$, it follows from 6.9 that $F_i^* k^1$ is well-defined.

Since $D(f_1, f_2)$ generates $H^n(P^n; Z_2)$, we can choose F_2 so that $D(F_1, F_2)$ is the generator ζ of $H^n(P^n \times o; Z_2) \subset H^n(P^n \times S^1; Z_2)$. Therefore by 6.9

$$F_1^* k^1 - F_2^* k^1 = d_1^{o,o}(0,\zeta) = w_1\langle\beta,\xi\rangle \cdot \zeta$$
$$= 0$$

since $w_1\langle\beta,\xi\rangle = 0$ from 6.2. Likewise $F_3^* k^1 = F_4^* k^1$. For a suitable choice of F_3, $D(F_1,F_3)$ is the generator η of $H^{n-1}(P^n \times o;Z) \subset H^{n-1}(P^n \times S^1;Z)$. Therefore by 6.9

$$F_1^* k^1 - F_3^* k^1 = d_1^{o,o}(\eta,0) = Sq^2\eta + w_2\langle\beta,\xi\rangle \cdot \eta .$$

This is the non-trivial element of $H^{n+1}(P^n \times S^1;Z_2)$: for $w_2\langle\beta,\xi\rangle = x \otimes \omega + x^2 \otimes 1$ from 6.2, where x and ω generate $H^1(P^n;Z_2)$ and $H^1(S^1;Z_2)$. Therefore either $F_1^* k^1 = F_2^* k^1 = 0$ and $F_3^* k^1 = F_4^* k^1 \neq 0$, or vice versa. Renumbering if necessary, we can assume the former. Then $F_1, F_2 : P^n \times S^1 \to Y^1$ lift to Y^2, whence by 6.4 (ii) β acts trivially on either f_{11} or f_{12}, and either f_{21} or f_{22}, and so trivially on all four. F_3 and F_4 do not lift, so $\beta \cdot \{f_{31}\} = \{f_{32}\}$, $\beta \cdot \{f_{41}\} = \{f_{42}\}$. Thus there are six orbits of the action of $\widetilde{KO}^{-1}(P^n)$ on $[X,Y^2]_f$. Since $BO_{n-1} \to Y^2$ is an $(n+1)$-equivalence, $[X,BO_{n-1}]_f$ has six $\widetilde{KO}^{-1}(P^n)$-orbits, and there are six bundles in the class ξ.

(d) $\underline{n = 4s+1,\ k = n-2}$; and

(e) $\underline{n = 4s+2,\ k = n-1.}$ The proofs are formally similar to that of (c), and are omitted.

(f) <u>$n = 4s+2$, $k = n-2$</u>. 5.12 shows that $[P^n, BO_{n-2}]_f$ contains two elements, so the number of orbits is one or two. There are two iff the corresponding $\langle \beta, \xi \rangle$ has geometrical dimension $\leq n-2$, but I do not know whether this is so.

Theorem 6.16 would be more interesting if one could give geometrical constructions for all the bundles, and more useful if one could find invariants distinguishing them.

Appendix 1. Proof of Theorem 3.3.

A1.1 Lemma. Let $F \to E \to B$ be a fibration such that

(i) F, E, B are all c-connected, where $c \geq 1$

(ii) $\pi_{2c}F \to \pi_{2c}E$ is a monomorphism.

Let Φ be the fibre of the suspended map $\Sigma E \to \Sigma B$. Then there is a natural diagram

$$\begin{tikzcd} \Sigma F \arrow[d, "\mu"'] \arrow[dr] & & \\ \Phi \arrow[r] & \Sigma E \arrow[r] & \Sigma B \end{tikzcd}$$

and μ induces isomorphisms of π_j for $j \leq 2c + 1$.

Proof. μ is induced by the canonical nullhomotopy of $\Sigma F \to \Sigma B$. The natural transformation of functors $\psi : 1 \to \Omega\Sigma$ gives a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\psi"'] & E \arrow[r] \arrow[dd, "\psi"] & B \arrow[dd, "\psi"] \\ \Omega\Sigma F \arrow[d, "\Omega\mu"'] \arrow[dr] & & \\ \Omega\Phi \arrow[r] & \Omega\Sigma E \arrow[r] & \Omega\Sigma B \end{tikzcd}$$

in which the rows are fibrations. Each ψ is a $(2c+1)$-equivalence by (i). Applying five-lemma arguments to the exact homotopy sequences of the rows, one sees that $\Omega\mu \circ \psi$ induces isomorphisms of π_j for $j \leq 2c$. Hence $\Omega\mu$ has the same property. The Lemma follows.

Proof of 3.3. Let F^i be the fibre of $X \to X^i$ in the tower $\mathcal{M}$, and M_i the fibre of $X^i \to B$. The fibre of $X^{i+1} \to X^i$ is $\Pi_{j\in Z} K(\Gamma^i{}_j, j)$. The diagram

$$\begin{array}{ccccccc}
F^{i+1} & \to & F^i & \to & F & \to & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Pi K(\Gamma^i{}_j, j) & \to & M_{i+1} & \to & X^{i+1} \\
 & & & & \downarrow & & \downarrow \\
 & & & & M_i & \to & X^i \\
 & & & & & & \downarrow \\
 & & & & & & B
\end{array}$$

contains fibrations $\Pi K(\Gamma^i{}_j, j) \to M_{i+1} \to M_i$ (1)

$F^i \to F \to M_i$ (2).

But (2) induces short exact sequences of π_r, because $\pi_r F^i \to \pi_r F$ is mono by the exact sequences of 2.9. Hence, by the diagram, (1) induces $0 \to \pi_r \Pi K(\Gamma^i{}_j, j) \to \pi_r M_{i+1} \to \pi_r M_i \to 0$. (3)

Since F is c-connected and $\mathcal{M}$ leaves $\pi_r F$ unaltered for $r > 2c$,

$\Gamma^i{}_j$ can be non-zero only in gradings $c < j \leq 2c$. (4)

Thus by (3), (4) and induction on i

$$\pi_j M_i \ \underline{\text{can be non-zero only for } c < j \leqslant 2c} \qquad (5)$$

We construct the tower $\sigma\mathcal{M}$ by induction. Suppose we have Y^r and f_r for $r \leqslant i$, and a diagram

$$\begin{tikzcd} & \sigma X \arrow[dl] \arrow[ddr] & \\ \sigma X^{i+1} \arrow[d] & & \\ \sigma X^i \arrow[rr, "f_i"] & & Y^i \end{tikzcd}$$

with f_i a $(2c+3)$-equivalence. Then σX, σX^{i+1}, σX^i, Y^i are fibrations over B with fibres ΣF, ΣM_{i+1}, ΣM_i, and N_i (say). Denote by Θ, Φ, Ψ, Δ respectively the fibres of $\sigma X \to Y^i$, $\sigma X^{i+1} \to \sigma X^i$, $\sigma X^{i+1} \to Y^i$ and $\sigma X \to \sigma X^i$. Then Φ is the fibre of $\Sigma M_{i+1} \to \Sigma M_i$, and Δ is the fibre of $\Sigma F \to \Sigma M_i$. We have diagrams

$$\begin{array}{ccccccc}
\Sigma[\Pi K(\Gamma^i_j,j)] & \xrightarrow{\mu} & \Phi & = & \Phi & \to & \Psi \\
& \searrow & \downarrow & & \downarrow & & \downarrow \\
& & \Sigma M_{i+1} & \subset & \sigma X^{i+1} & = & \sigma X^{i+1} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Sigma M_i & \subset & \sigma X^i & \xrightarrow{f_i} & Y^i
\end{array} \qquad (6)$$

$$\begin{array}{ccccccc}
\Sigma F^i & \xrightarrow{\mu} & \Delta & = & \Delta & \to & \Theta \\
& \searrow & \downarrow & & \downarrow & & \downarrow \\
& & \Sigma F & \subset & \sigma X & = & \sigma X \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Sigma M_i & \subset & \sigma X^i & \xrightarrow{f_i} & Y^i
\end{array} \qquad (6')$$

in which each column is a fibration: the maps μ arise from A1.1 and induce isomorphisms of π_r for $r \leqslant 2c + 1$. Since fi is a (2c+3) equivalence, the homotopy exact sequences imply that $\Phi \to \Psi$, $\Delta \to \Theta$ are (2c+2)-equivalences. Hence <u>the top rows of (6), (6') induce isomorphisms on π_r and H^r for $r \leqslant 2c + 1$</u>. (7)

<u>Construction of Y^{i+1}</u>. Consider (6). By (7), the suspended fundamental class $\Sigma\iota \in \mathcal{H}^1(\Sigma[\Pi K(\Gamma^i_j, j)];\Gamma^i)$ comes from unique class $\alpha_\phi \in \mathcal{H}^1(\Phi; \Gamma^i)$, $\alpha_\psi \in \mathcal{H}^1(\Psi; \Gamma^i)$. Since $-\iota \in \mathcal{H}^0(\Pi K(\Gamma^i{}_j,j);\Gamma^i)$ transgresses to the k-invariant k^i in the fibration $\Pi K(\Gamma^i{}_j,j) \to X^{i+1} \to X^i$,

it follows that α_ϕ transgresses to Sk^i in the centre column of (6), where S is the suspension homomorphism of 3.1. Let $\kappa^i \in \mathcal{H}^2(Y^i;\Gamma^i)$ be the unique element with $f_i^*\kappa^i = -Sk^i$ (f_i is a (2c+3)-equivalence). Then α_ψ transgresses to $-\kappa^i$ in the right-hand column, and by 2.7 we construct

$$\begin{array}{ccc} \Psi & \xrightarrow{\alpha_\psi} & \Pi K(\Gamma^i{}_j, j+1) \\ \downarrow & & \downarrow \\ \sigma X^{i+1} & \rightarrow & P_B\mathcal{K}(\Gamma^i,2) \\ \downarrow & & \downarrow \\ Y^i & \xrightarrow{\kappa^i} & \mathcal{K}(\Gamma^i,2) \end{array} \qquad (8)$$

Let $Y^{i+1} \rightarrow Y^i$ be the fibration induced over Y^i by κ^i, and $f_{i+1} : \sigma X^{i+1} \rightarrow Y^{i+1}$ the natural map. Then there is a commutative diagram

$$\begin{tikzcd} & \sigma X \arrow[dl] \arrow[dr] & \\ \sigma X^{i+1} \arrow[rr, "f_{i+1}"] \arrow[d] & & Y^{i+1} \arrow[d] \\ \sigma X^i \arrow[rr, "f_i"] & & Y^i \end{tikzcd}$$

<u>$\sigma X \rightarrow Y^{i+1} \rightarrow Y^i$ is an elementary factorisation</u>. To check this we have to show then induced map on fibres, $\Theta \rightarrow \Pi K(\Gamma^i{}_j, j+1)$, induces epimorphisms on π_*. The map $\sigma X \rightarrow \sigma X^{i+1}$ induces a map from diagram

(6') to (6) : the composites of the two top rows appear in

$$\begin{array}{ccccc} \Sigma F^i & \rightarrow & \Theta & & \\ \downarrow & & \downarrow & & \\ \Sigma[\Pi K(\Gamma^i_j, j)] & \underset{\nu}{\rightarrow} & \Psi & \xrightarrow{\alpha_\psi} & \Pi K(\Gamma^i_j, j+1) \end{array} \qquad (9)$$

Since $\mathcal{M}$ is an MPT, $F^i \rightarrow \Pi K(\Gamma^i_j, j)$ induces epimorphisms of π_*; so by the suspension theorem $\Sigma F^i \rightarrow \Sigma[\Pi K(\Gamma^i_j, j)]$ induces epimorphisms of π_r, $r \leq 2c + 2$. The bottom row induces epimorphisms of π_* because it represents the suspended fundamental class. Hence by commutativity $\Theta \rightarrow \Psi \rightarrow \Pi K(\Gamma^i_j, j+1)$ induces epimorphisms of π_r, $r \leq 2c + 2$: and $\Pi K(\Gamma^i_j, j+1)$ has no homotopy outside this range.

Thus we may take $Y^{i+1} \rightarrow Y^i$ as the next stage of $\sigma\mathcal{M}$.

f_{i+1} <u>is a (2c+3)-equivalence</u>. Since $Y^{i+1} \rightarrow Y^i$ is the induced fibration in (8), it suffices to show α_ψ $\Psi \rightarrow \Pi K(\Gamma^i_j, j+1)$ is a (2c+3)-equivalence. Consider (9). ν induces isomorphisms of π_r for $r \leq 2c + 1$ by (6), and the natural map $\alpha_\psi \circ \nu$ also does. So it remains to investigate $\alpha_{\psi *}$ on π_{2c+2} and π_{2c+3}. These homotopy groups of $\Pi K(\Gamma^i_j, j+1)$ are zero, so we need only show $\pi_{2c+2} \Psi = 0$. But Ψ is the fibre of $\sigma X^{i+1} \rightarrow Y^i$, and hence (restricting to fibres over B) Ψ is the fibre of $\Sigma M_{i+1} \rightarrow N_i$. Thus

$\pi_{2c+3}N_i \to \pi_{2c+2}\Psi \to \pi_{2c+2}\Sigma M_{i+1}$ is exact. But $\pi_{2c+2}\Sigma M_{i+1} = 0$ by (5) and the suspension theorem, and $\pi_{2c+3}N_i = 0$ from the analogue of (5) for the tower $\sigma\mathcal{M}$.

The inductive step is now complete, and (i), (iii) of 3.3 are proved. We prove (ii). $\sigma\mathcal{M}$ leaves $\pi_j(\Sigma F)$ unaltered for $j > 2c + 1$ by the choice of coefficient modules (4). Since kills homotopy up to $\pi_{2c}F$, F^n is 2c-connected; hence ΣF^n is (2c+1)-connected. Take $i = n$ in (6'): by (7), the fibre of $\sigma X \to Y^n$ is (2c+1)-connected. Hence $\sigma\mathcal{M}$ kills homotopy up to $\pi_{2c+1}(\Sigma F)$.

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